Spherical polygons and differential equations

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August 15, 2020

Abstract

This is an exposition of some results on classification of spherical polygons with prescribed interior angles and prescribed images of vertices under a conformal map onto the unit disk.

MSC 2010: 30C20, 34M03.

Keywords: surfaces of positive curvature, conic singularities, Schwarz equation, accessory parameters, conformal mapping, circular polygons, Kostka numbers.

This paper contains an exposition of the results from [5], [6] and [7].

A *polygon* is a surface homeomorphic to the closed disk, with several marked points on the boundary called *corners*, equipped with a Riemannian metric of constant curvature K, such that the *sides* (arcs between the corners) are geodesic, and the metric has *conical singularities* at the corners.

A conical singularity is a point near which the length element of the metric is

$$ds = \frac{2\alpha|z|^{\alpha-1}|dz|}{1+K|z|^2},$$

where z is a local conformal coordinate. The number $2\pi\alpha > 0$ is the angle at the conical singularity. The interior angle of our polygon is $\pi\alpha$ radians. *We prefer to measure angles in half-turns, so in what follows, "integer angle" will mean that α is an integer.* These angles can be arbitrarily large. Every polygon can be mapped conformally onto the unit disk. We consider the

[*]Supported by NSF grant DMS-1351836

[†]Supported by NSF grant DMS-1161629

problem of classification up to isometry of polygons with prescribed angles and prescribed corners.

By *prescribed corners* we mean that the images of the corners on the unit circle under the conformal map of the polygon onto the unit disk are prescribed.

The problem becomes simpler if we consider marked polygons: the corners are marked as $a_0, a_1 \dots, a_{n-1}$ in the order of positive orientation of the boundary, and two polygons are considered equal if there exists an isometry between them which sends a_j to a'_j, $0 \leq j \leq n-1$. We consider only marked polygons.

Flat polygons, $K = 0$. The angles must satisfy the condition

$$\sum \alpha_j = n - 2.$$

For any given angles and prescribed corners, there exists a polygon, which is unique up to scaling.

Proof: Christoffel–Schwarz formula.

Hyperbolic polygons, $K < 0$. The angles must satisfy

$$\sum \alpha_j < n - 2.$$

For any given angles and prescribed corners, there exists a unique polygon (E. Picard [15, 16, 17, 18], M. Heins [10], M. Troyanov [20]).

We study *spherical polygons*, assuming $K = 1$. The necessary condition on the angles,

$$\sum \alpha_j > n - 2,$$

follows from the Gauss–Bonnet formula. If the angles are *sufficiently small*, $0 < \alpha_j < 1$, then we have the necessary and sufficient condition

$$0 < \sum(\alpha_j - 1) + 2 < 2 \min \alpha_j,$$

proved by M. Troyanov [20], and uniqueness for this case was proved by F. Luo and G. Tian [12]).

Spherical triangles were classified by F. Klein [11], A. Eremenko [2], S. Fujimori, Y. Kawakami, M. Kokubu, W. Rossman, M. Umehara and K. Yamada [8].

If all α_j are not integers, the necessary and sufficient condition for the existence of a spherical triangle is

$$\cos^2 \pi\alpha_0 + \cos^2 \pi\alpha_1 + \cos^2 \pi\alpha_2 + 2\cos\pi\alpha_0 \cos\pi\alpha_1 \cos\pi\alpha_2 < 1,$$

and the triangle is unique.

If α_0 is an integer but α_1 and α_2 are not, then the necessary and sufficient condition is that either $\alpha_1 + \alpha_2$ or $\alpha_1 - \alpha_2$ is an integer $m < \alpha_0$, with m and α_0 of opposite parity.

The triangle with an integer corner is not unique: there is a 1-parametric family when only one angle is integer, and a 2-parametric family when all angles are integer.

Polygons with only one non-integer angle do not exist.

Developing map. A surface D of constant curvature 1 is locally isometric to the standard sphere $\mathbf{S}$. This isometry is conformal, has an analytic continuation to the whole polygon, and is called the *developing map* $f : D \to \mathbf{S}$ [2], [1].

We say that spherical polygons are *equivalent* if their developing maps differ by a post-composition with an element of $PSL(2, \mathbf{C})$ acting as fractional-linear transformations of the sphere.

Let us choose the upper half-plane H as the conformal model of our polygon, with n corners $a_0, \ldots, a_{n-1}$, and choose $a_{n-1} = \infty$. Accordingly, we sometimes denote α_{n-1} as α_∞. The other corners are real points. Then $f : H \to \mathbf{S}$ is a meromorphic function mapping the sides into great circles. By the Symmetry Principle, f has an analytic continuation to a multi-valued function in $\overline{\mathbf{C}} \setminus \{a_0, \ldots, a_{n-1}\}$ whose monodromy is a subgroup of $PSU(2) \sim SO(3)$ (acting by isometries of the sphere).

Such a function must be a ratio of two linearly independent solutions of the Fuchsian differential equation

$$w'' + \sum_{k=0}^{n-2} \frac{1 - \alpha_k}{z - a_k} w' + \frac{P(z)}{\prod_{k=0}^{n-2}(z - a_k)} w = 0,$$

where P is a real polynomial of degree $n - 3$ whose top coefficient can be expressed in terms of the α_j. The remaining $n - 3$ coefficients of P are called the *accessory parameters*. The monodromy group of this equation must be conjugate to a subgroup of $PSU(2)$. In the opposite direction, if a

Fuchsian differential equation with real singularities and real coefficients has the monodromy group conjugate to a subgroup of $PSU(2)$, then the ratio of two linearly independent solutions restricted to H is a developing map of a spherical polygon.

Thus classification of spherical polygons with given angles and corners is equivalent to the following problem:

For a Fuchsian equation with given real parameters a_j, α_j, to find the real values of accessory parameters for which the monodromy group of that equation is conjugate to a subgroup of $PSU(2)$. These values of accessory parameters are in bijective correspondence with the equivalence classes of spherical polygons.

Spherical polygons with all integer angles. In this case, the developing map is a real rational function with real critical points. The multiplicities of the critical points are $\alpha_j - 1$. Such functions have been studied in great detail (A. Eremenko and A. Gabrielov [3], I. Scherbak [21], A. Eremenko, A. Gabrielov, M. Shapiro, F. Vainshtein [4].)

The necessary and sufficient condition on the angles is $\sum(\alpha_j - 1) = 2d - 2$, where $d = \deg f$ is an integer, and $\alpha_j \leq d$ for all j. For given angles, there exist exactly $K(\alpha_0 - 1, \ldots, \alpha_{n-1} - 1)$ of the equivalence classes of polygons, where K is the *Kostka number*: it is the number of ways to fill in a table with two rows of length $d - 1$ with $\alpha_0 - 1$ zeros, $\alpha_1 - 1$ ones, *etc.*, so that the entries are non-decreasing in the rows and increasing in the columns.

Polygons with two non-integer angles. Let α_0 and α_{n-1} be non-integer, while the rest of the angles α_j are integer. *We do not assume here that the order $\alpha_0, \ldots, \alpha_{n-1}$ corresponds to the positive orientation.*

Assuming $a_0 = 0$ and $a_{n-1} = \infty$ we conclude that the developing map has the form

$$f(z) = z^\alpha \, \frac{P(z)}{Q(z)},$$

where $\alpha \in (0, 1)$ and P, Q are real polynomials without common factors. For this case, a necessary and sufficient condition on the angles is the following

Theorem 1. *Let $\sigma := \alpha_1 + \ldots + \alpha_{n-2} - n + 2$.*
a) If $\sigma + [\alpha_0] + [\alpha_{n-1}]$ is even, then $\alpha_0 - \alpha_{n-1}$ is an integer of the same parity as σ, and $|\alpha_0 - \alpha_{n-1}| \leq \sigma$.
b) If $\sigma + [\alpha_0] + [\alpha_{n-1}]$ is odd, then $\alpha_0 + \alpha_{n-1}$ is an integer of the same parity as σ, and $\alpha_0 + \alpha_{n-1} \leq \sigma$.

Finding all polygons with prescribed angles is equivalent in this case to solving the equation

$$z(P'Q - PQ') + \alpha PQ = R$$

with respect to real polynomials P and Q of degrees p and q, respectively, where R is a given real polynomial of degree $p + q$. The map

$$W_\alpha : (P, Q) \mapsto z(P'Q - PQ') + \alpha PQ$$

is finite and its degree equals equals

$$\binom{p+q}{p}$$

(it is a linear projection of a Veronese variety), and one can show that when all roots of R are non-negative, all solutions $(P, Q) \in W_\alpha^{-1}(R)$ are real.

Enumeration of polygons with two adjacent non-integer angles. An important special case is when a_0 and a_{n-1} are adjacent corners of the polygon, $2\alpha_0$ and $2\alpha_{n-1}$ are odd integers, while all other α_j are integers. Equivalence classes of such polygons are in bijective correspondence with *odd real rational functions with all critical points real*, given by

$$f(z) = g(\sqrt{z}),$$

where f is the developing map of our polygon and g is a rational function as above. By a deformation argument, this gives the following

Theorem 2. *If the angles satisfy the necessary and sufficient condition given above, and the corners $a_0 = 0$ and $a_{n-1} = \infty$ are adjacent, then there are exactly*

$$E(2\alpha_0 - 1, \alpha_1 - 1, \ldots, \alpha_{n-2} - 1, 2\alpha_{n-1} - 1)$$

equivalence classes of polygons, where $E(m_0, \ldots, m_{n-1})$ is the number of chord diagrams in H, symmetric with respect to $z \mapsto -\overline{z}$, with the vertices $0 = a_0 < a_1 < \ldots < a_{n-2} < a_{n-1} = \infty$ and $-a_1, \ldots, -a_{n-2}$, and m_j chords ending at each vertex a_j.

If a_0 and a_{n-1} are not adjacent, E gives an upper bound on the number of equivalence classes of polygons.

One can express E in terms of the Kostka numbers.

Proposition. *Let m_0 and m_{n-1} be even. Then*

$$E(m_0, m_1, \ldots, m_{n-2}, m_{n-1}) = K(r, m_1, \ldots, m_{n-2}, s),$$

where positive integers r and s satisfy

$$r + s > m_1 + \ldots + m_{n-2}, \tag{1}$$

and can be defined as follows:
If $\mu := (m_0 + m_{n-1})/2 + m_1 + \ldots + m_{n-2}$ is even, then $r = m_0/2 + k,\ s = m_{n-1}/2 + k$, where k is large enough, so that (1) is satisfied.
If μ is odd, then $r = (m_0 + m_{n-1})/2 + k + 1,\ s = k$, and k is large enough, so that (1) is satisfied.

Spherical polygons with 3 non-integer angles. In this case, the images of the sides under the developing map are contained in three circles. The intersection of these three circles may consist of two points, and this case is called *exceptional*. In the exceptional case, the three circles are equivalent to three lines intersecting at one finite point.

Theorem. *Let Q be a circular polygon with non-integer angles θ, θ' and θ'' and the rest of the angles integers. Suppose that the images of the sides under the developing map are not tangent to each other. Then Q is equivalent to a spherical polygon if and only if it is either exceptional or*

$$\cos^2 \pi\theta + \cos^2 \pi\theta' + \cos^2 \pi\theta'' + 2(-1)^\sigma \cos \pi\theta \cos \pi\theta' \cos \pi\theta'' < 1,$$

where

$$\sigma = \sum_{j:\alpha_j \in \mathbf{Z}} (\alpha_j - 1).$$

Spherical quadrilaterals. Heun's equation. In the case $n = 4$ the Fuchsian equation for the developing map is the Heun's equation

$$w'' + \left(\frac{1-\alpha_0}{z} + \frac{1-\alpha_1}{z-1} + \frac{1-\alpha_2}{z-a} \right) w' + \frac{Az - \lambda}{z(z-1)(z-a)} w = 0,$$

where A can be expressed in terms of α_j, and λ is the accessory parameter.

We can place three singularities at arbitrary points, so we choose $a_0 = 0$, $a_1 = 1$, $a_2 = a$, $a_3 = \infty$.

The condition that the monodromy belongs to $PSU(2)$ is equivalent to an equation of the form $F(a, \lambda) = 0$. This equation is algebraic if at least one angle is integer.

Theorem 2 in the case of quadrilaterals with two integer and two non-integer angles specializes to the following

Theorem 3. *The number of classes of quadrilaterals with two integer and two non-integer angles is at most*

$$\min\{\alpha_1, \alpha_2, k+1\},$$

where

$$k+1 = \begin{cases} (\alpha_1 + \alpha_2 - |\alpha_0 - \alpha_3|)/2 & \textit{in case a)} \\ (\alpha_1 + \alpha_2 - \alpha_0 - \alpha_3)/2 & \textit{in case b).} \end{cases}$$

If $a > 0$ we have equality.

Here cases a) (when $\alpha_0 - \alpha_3$ is integer) and b) (when $\alpha_0 + \alpha_3$ is integer) are as in Theorem 1. Condition $a > 0$ means that the corners a_1 and a_2 with integer angles are adjacent.

Quadrilaterals with non-adjacent integer angles. Let $\delta = \max(0, \alpha_1 + \alpha_2 - [\alpha_0] - [\alpha_3])/2$.

Theorem 4. *The number of equivalence classes of quadrilaterals with non-adjacent corners a_1 and a_2, with integer angles α_1 and α_2, is at least*

$$\min\{\alpha_1, \alpha_2, k+1\} - 2\left[\frac{1}{2}\min\{\alpha_1, \alpha_2, \delta\}\right], \tag{2}$$

where k is the same as in Theorem 3.

Notice that in case b) of Theorems 1 and 3, the lower bound (2) becomes 0 when $\min\{\alpha_1, \alpha_2, k+1\}$ is even and 1 if $\min\{\alpha_1, \alpha_2, k+1\}$ is odd.

Quadrilaterals with three non-integer corners. Let α_0 be the integer angle, and $\alpha_1, \alpha_2, \alpha_3$ non-integer angles. In the exceptional case, the condition

$$\cos\pi\frac{\alpha_1 + \alpha_2 + \theta_3}{2} \cos\pi\frac{-\alpha_1 + \alpha_2 + \alpha_3}{2} \cos\pi\frac{\alpha_1 - \alpha_2 + \alpha_3}{2} \cos\pi\frac{\alpha_1 + \alpha_2 - \alpha_3}{2} = 0$$

must be satisfied, and for all sets of angles satisfying this condition, there is only a finite set of possible moduli a for which quadrilaterals exist.

Theorem. *The number of quadrilaterals with integer* α_0 *and non-integer* $\alpha_1, \alpha_2, \alpha_2$ *and prescribed modulus* a *is at most* $\alpha_0 - 1$ *and at least*

$$\alpha_0 - 2\left[\min\left(\frac{\alpha_0}{2}, \frac{1+[\alpha_2]}{2}, \frac{\delta}{2}\right)\right],$$

where $\delta = \max(0, 1 + [\alpha_2] + \alpha_0 - [\alpha_2] - [\alpha_3])/2$.

The upper estimate is exact, and we conjecture that the lower estimate is exact as well.

Algebraic method. In the case of quadrilaterals with one or two integer angles, our problem is equivalent to counting real solutions of an algebraic equation $F(a, \lambda) = 0$, expressing the fact that the Heun's equation has $PSU(2)$ monodromy. Degree of this polynomial with respect to λ gives the upper bound on the number of equivalence classes of quadrilaterals. The polynomial F is the spectral determinant of an eigenvalue problem for a certain finite Jacobi matrix. To see this, we re-write the Heun's equation as an eigenvalue problem

$$z(z-1)(z-a)\left(w'' + \left(\frac{1-\alpha_0}{z} + \frac{1-\alpha_2}{z-1} + \frac{1-\alpha_2}{z-a}\right)w'\right) + Azw = \lambda w.$$

The operator in the left-hand side can be represented by a Jacobi matrix acting on the sequence of the Taylor coefficients of w.

In the cases we consider, the problem can be reduced to the existence of a finite-dimensional eigenvector.

There is a natural quadratic form with respect to which the Jacobi matrix is symmetric [9]. This quadratic form is positive definite in the case when the corners with integer angles are adjacent, [14]. In the case when they are not adjacent we use Pontryagin's theorem [19] on the matrices symmetric with respect to an indefinite form [13]. This method seems to work only in the cases when the eigenvalue problem is finite-dimensional, that is the equation $F(a, \lambda) = 0$ is algebraic.

Geometric method. The developing map is a local homeomorphism, except at the corners, of a closed disk D to the standard sphere $\mathbf{S}$. The sides are mapped to great circles. These great circles define a partition (cell decomposition) of the sphere. Taking the f-preimage of this partition, and adding vertices corresponding to the integer corners, we obtain a cell decomposition of D which is called a *net*. Two nets are considered equivalent if they can be mapped to each other by an orientation-preserving homeomorphism of the disk, respecting labeling of the corners.

It is easy to see that a net, together with the partition of the sphere by the great circles, define the polygon us to an isometry. So the problems of existence of polygons are reduced in principle to classification and counting the nets, which is a combinatorial problem.

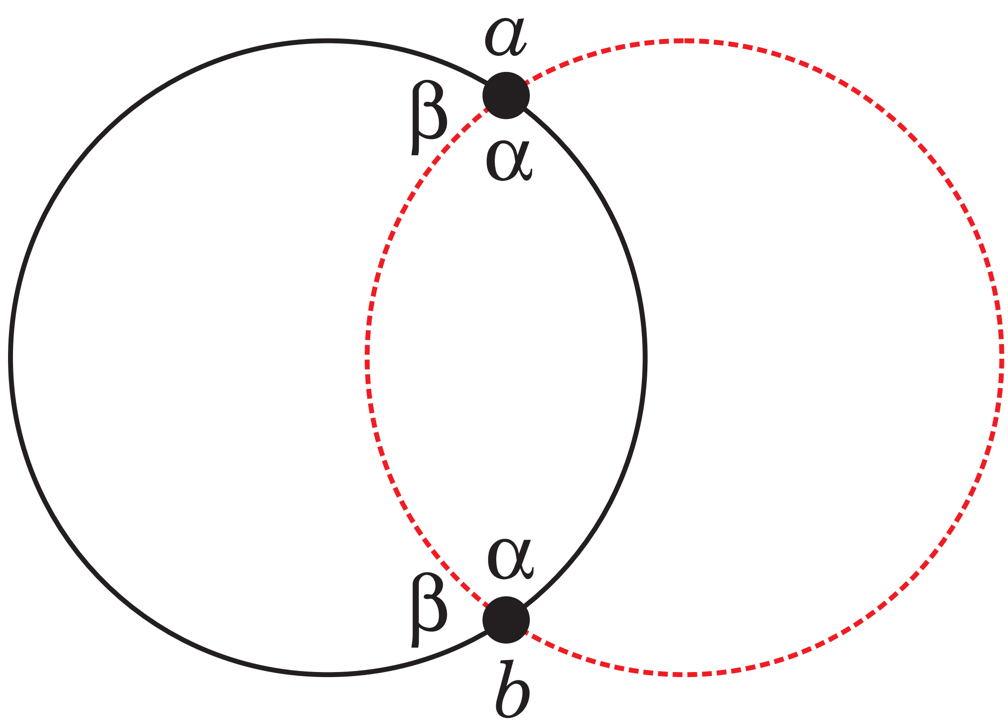

Fig. 1. Partition of the Riemann sphere by two great circles.

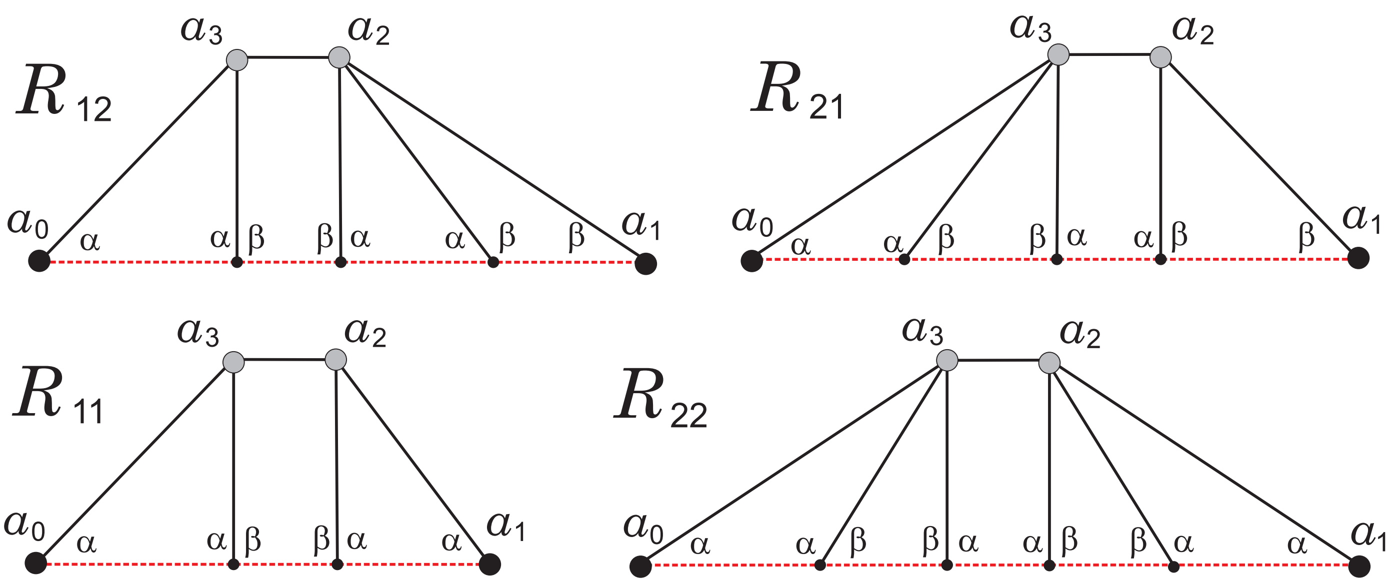

Fig. 2. Nets with two adjacent integer corners.

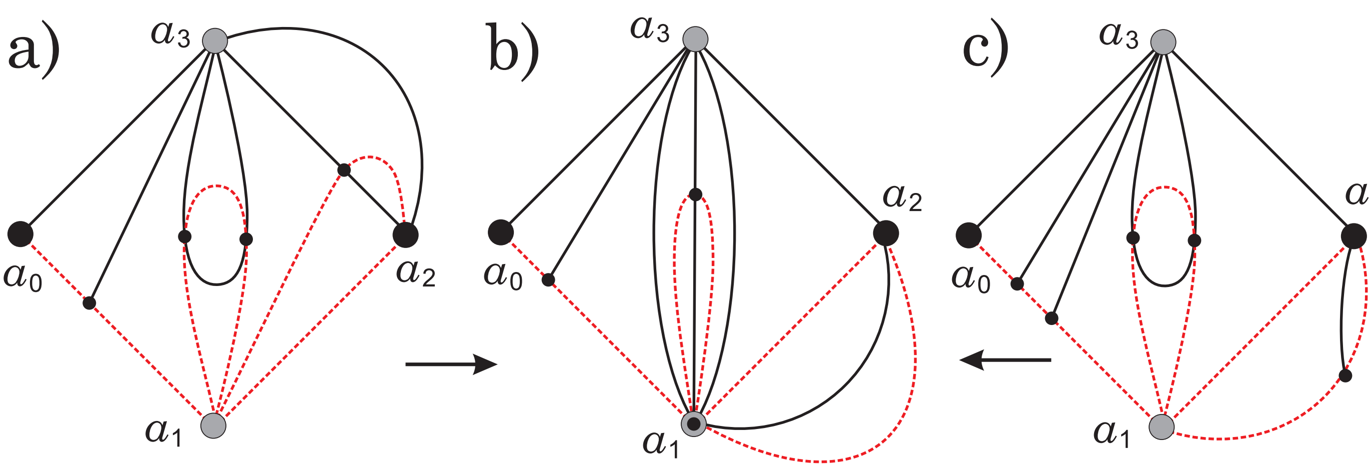

Fig. 3. Chain of nets with two opposite integer corners.

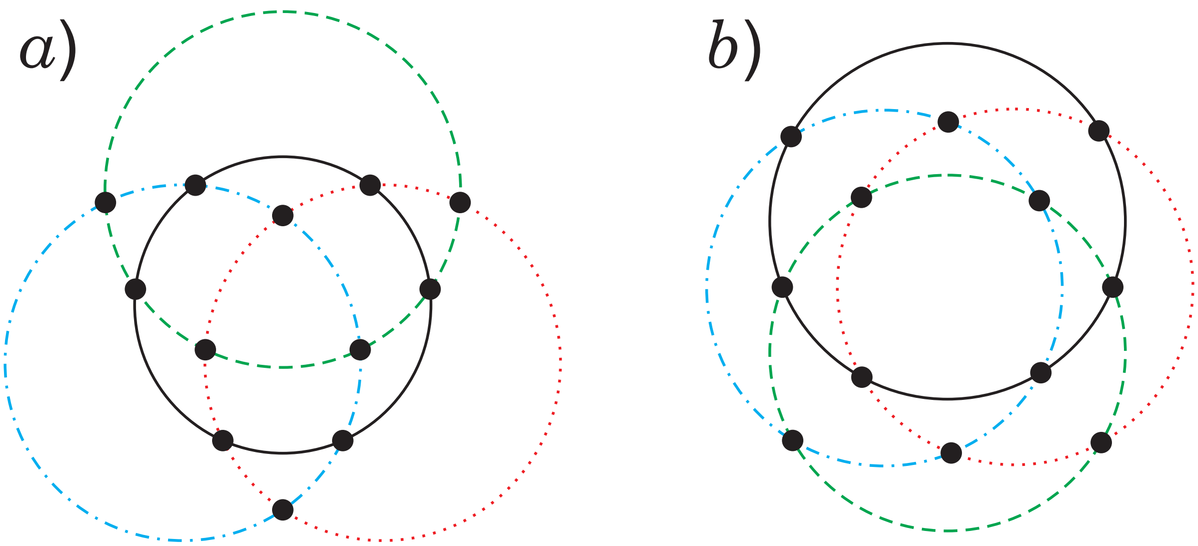

Fig. 4. Partition of the Riemann sphere by four great circles (two views).

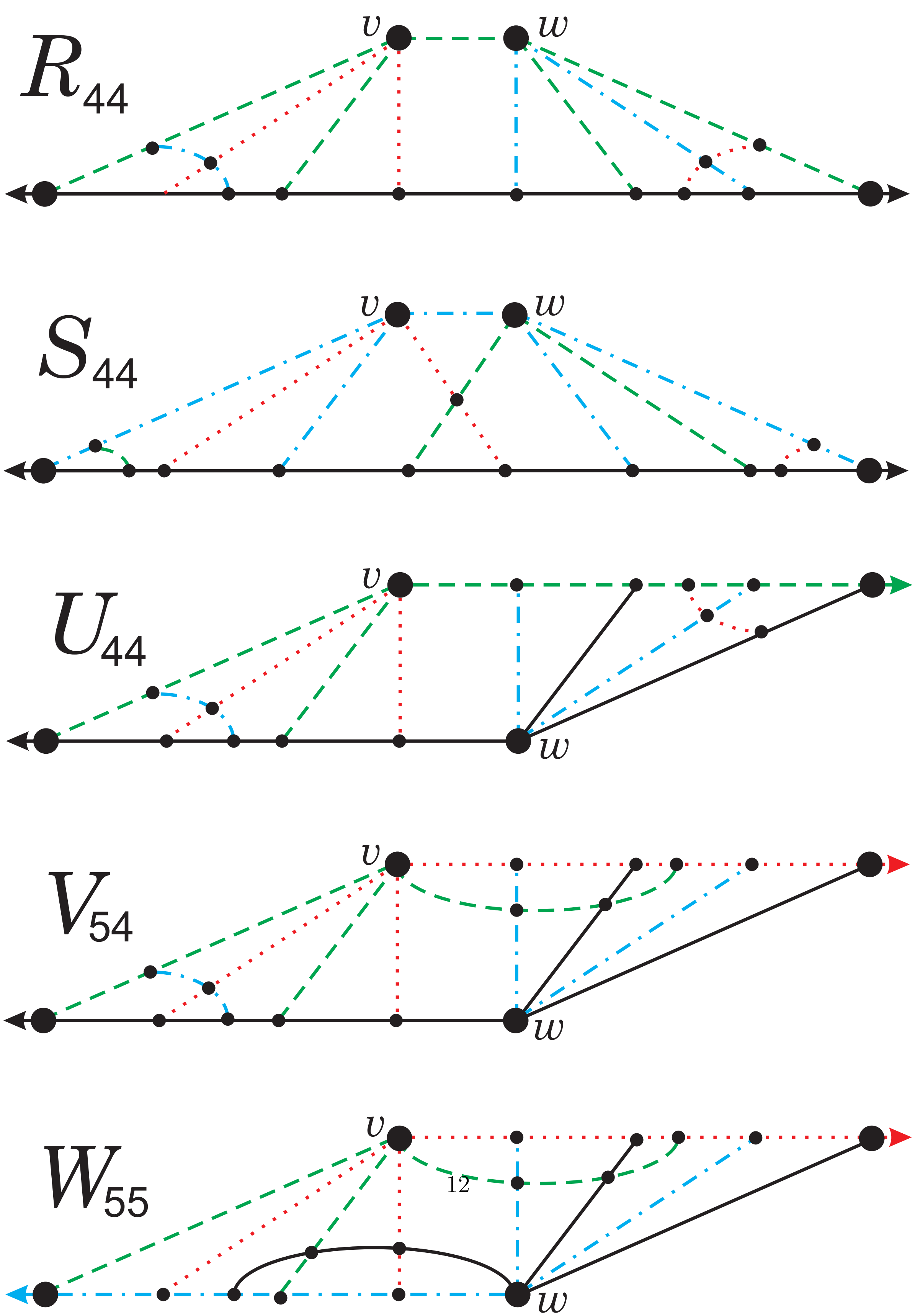

Fig. 5. Nets with four non-integer corners.

Our strategy is the following. First we classify all possible nets with given angles. Then we construct certain curves in the space of quadrilaterals with given angles, by moving the images of integer corners along the circles of the partition of the sphere **S**, and keeping the net fixed.

In the "good case" when the corners with integer angles are adjacent, we can show that the conformal modulus of the quadrilateral tends to 0 and ∞ on the ends of the curve. This proves the existence of a quadrilateral with prescribed angles and prescribed modulus. In the "difficult case" when the corners with integer angles are not adjacent, to construct the curves on which the modulus changes from 0 to ∞, one needs sometimes to paste together several curves with fixed nets.

The method is applicable, in principle, to all cases, no matter whether the accessory parameter problem is algebraic or not, but the computations become more complicated as the partition of the sphere by great circles contains more circles.

In the following pictures we choose the upper half-plane conformal model with corners $0, 1, a, \infty$, integer angles α_1 and α_2 at 0 and 1, non-integer angles α_0 and α_3 at a and ∞, and plot the algebraic function $\lambda(a)$ which is defined by the condition that the monodromy of the Heun's equation is unitary. The values $0 < a < 1$ correspond to quadrilaterals with opposite integer corners.

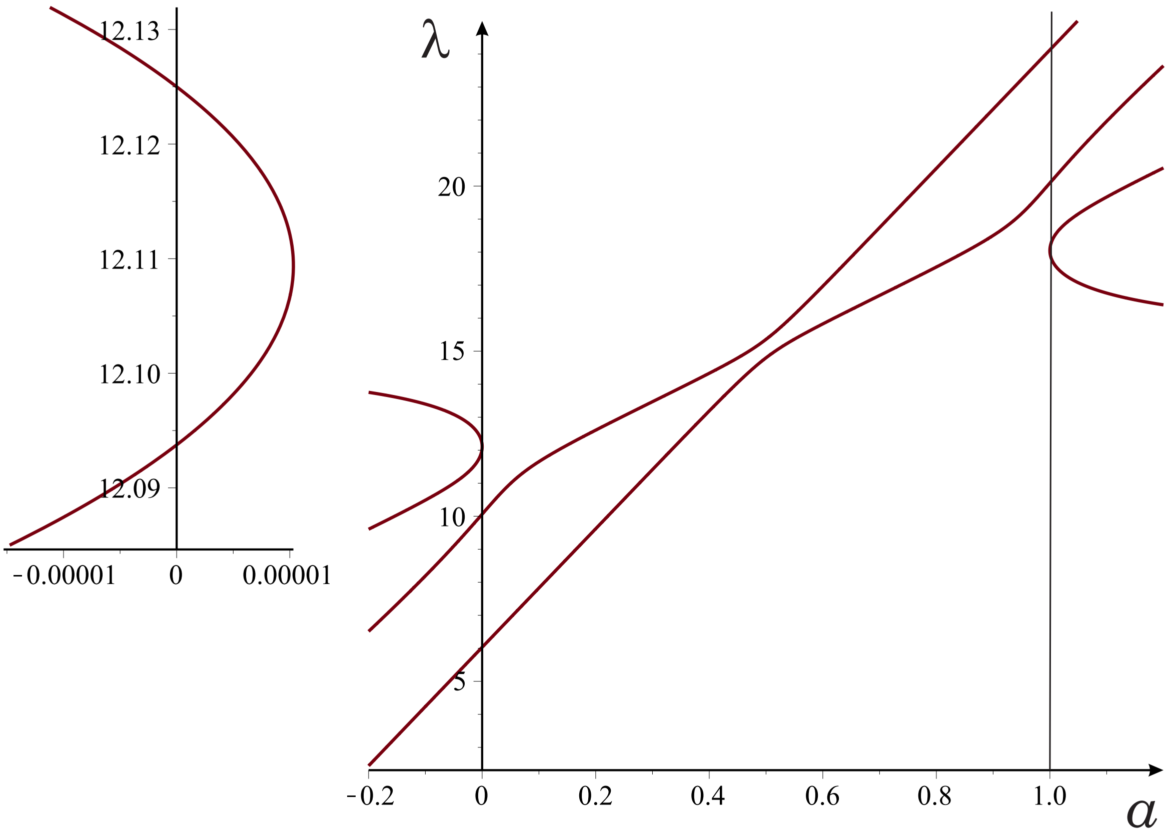

Fig. 6. $\alpha_1 = 6,\ \alpha_2 = 4,\ \alpha_0 = \alpha_3 = 65/32$

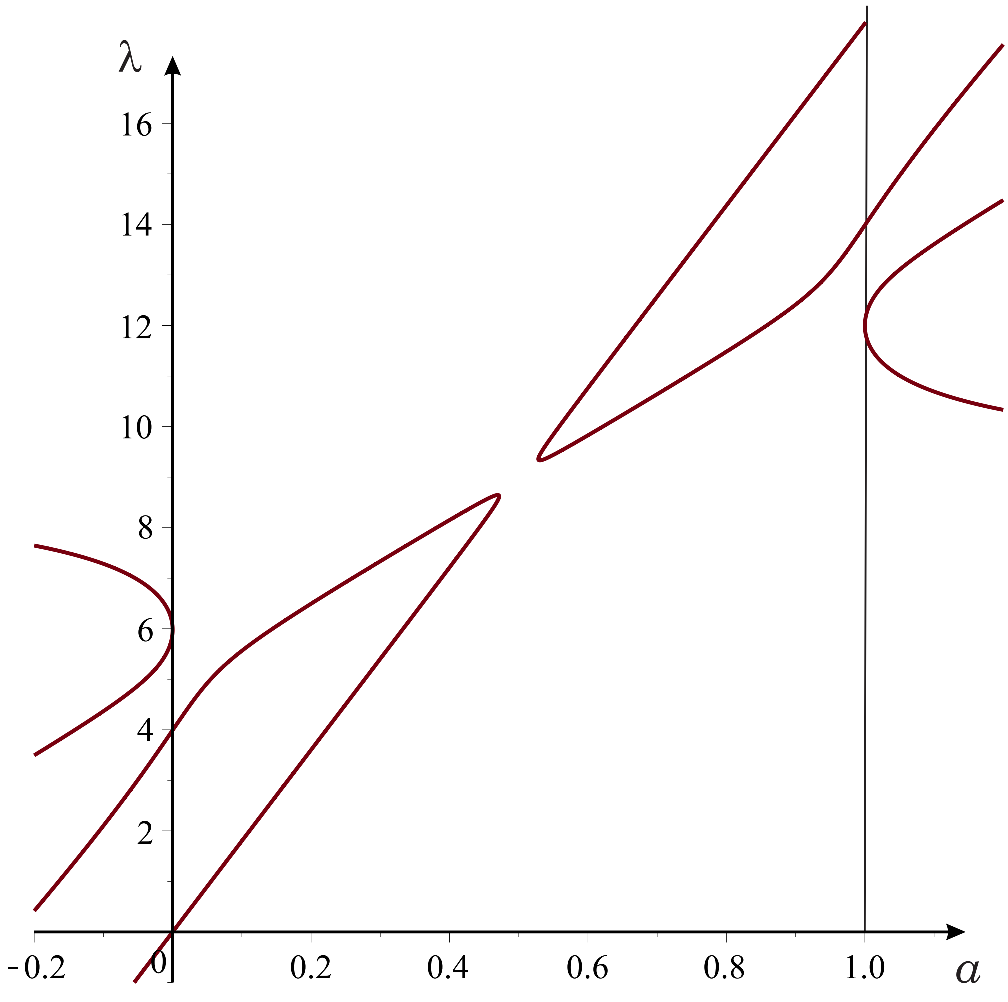

Fig. 7. $\alpha_1 = 6,\ \alpha_2 = 4,\ \alpha_0 = \alpha_3 = 255/128$

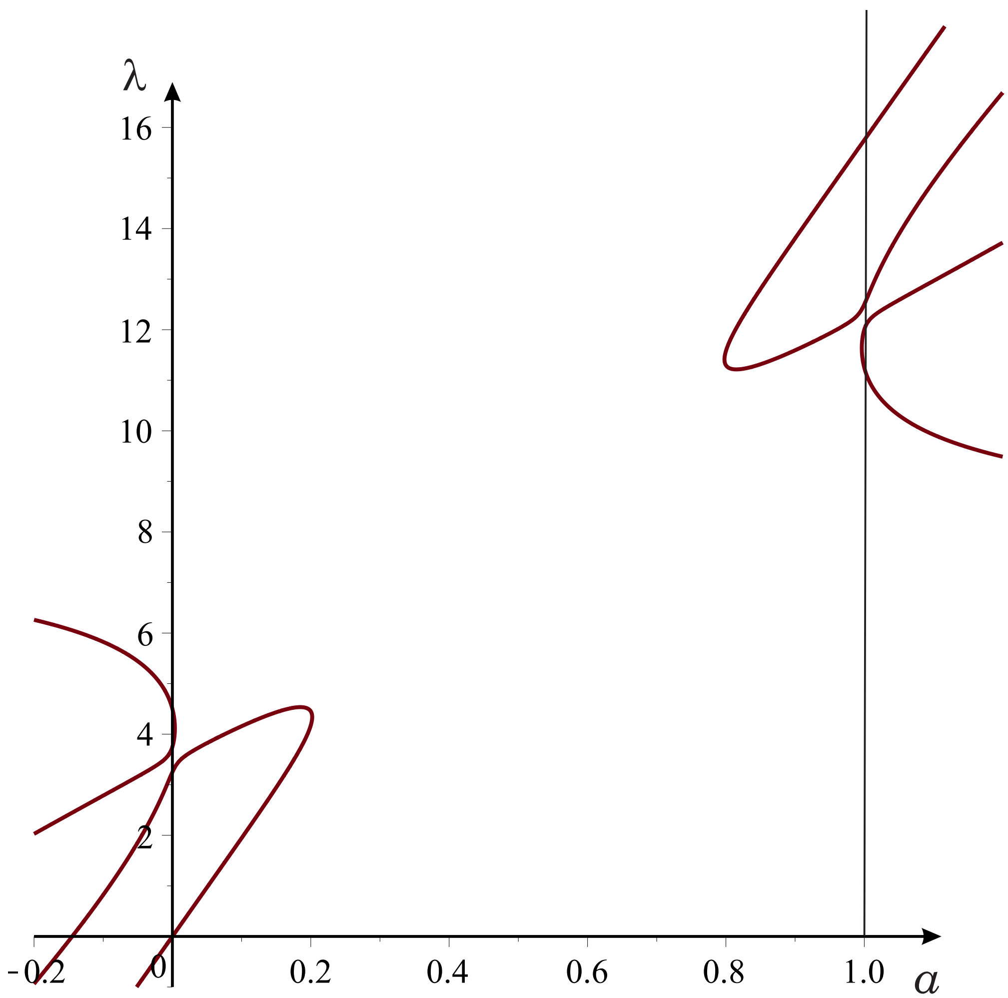

Fig. 8. $\alpha_1 = 6,\ \alpha_2 = 4,\ \alpha_0 = \alpha_3 = 5/4$

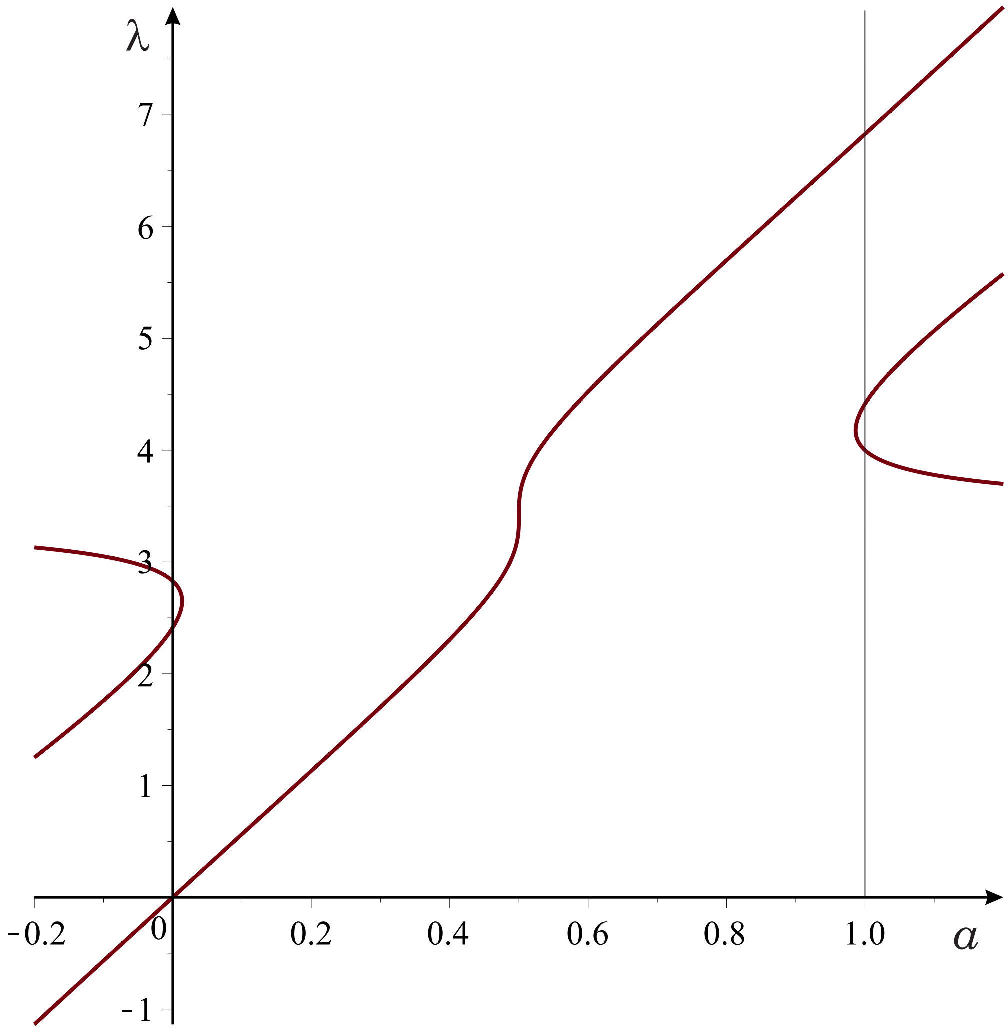

Fig. 9. $\alpha_1 = \alpha_2 = 3,\ \alpha_0 = \alpha_3 = \sqrt{2}$

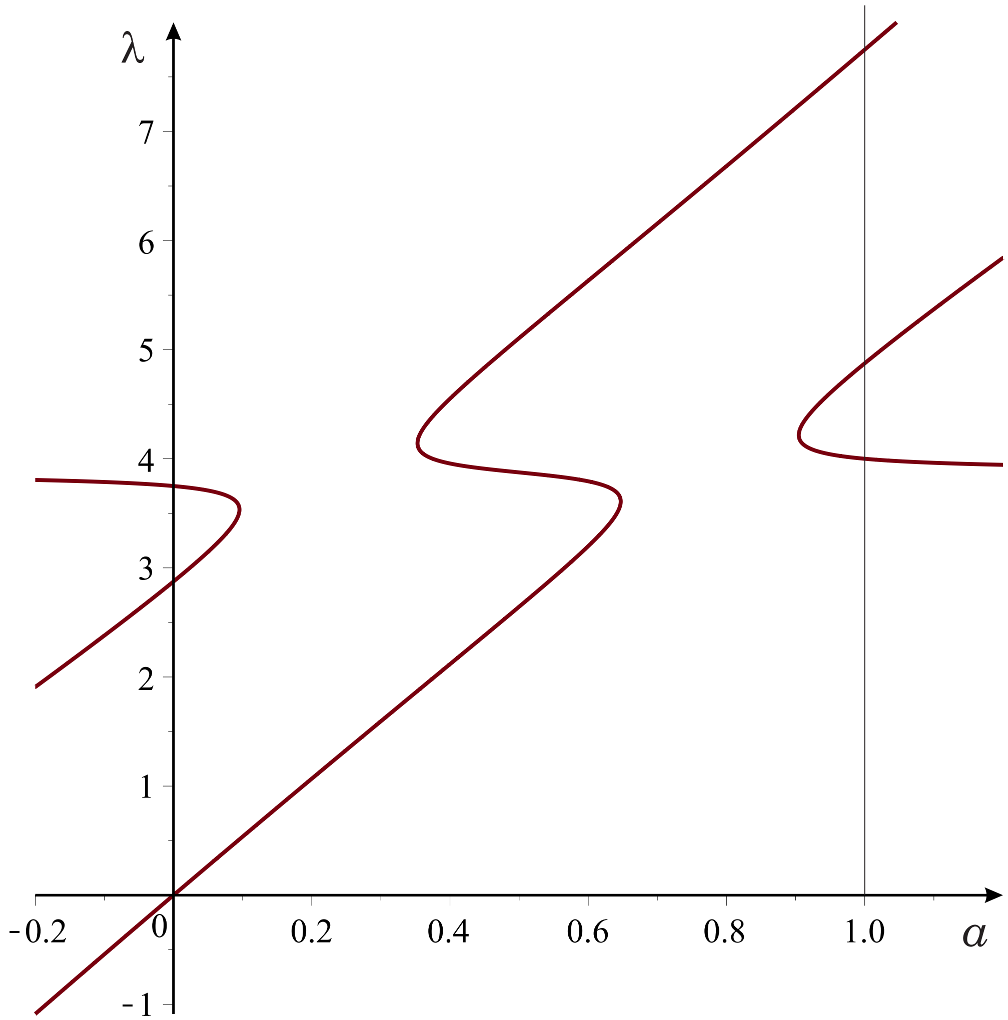

Fig. 10. $\alpha_1 = \alpha_2 = 3,\ \alpha_0 = \alpha_3 = 15/8$

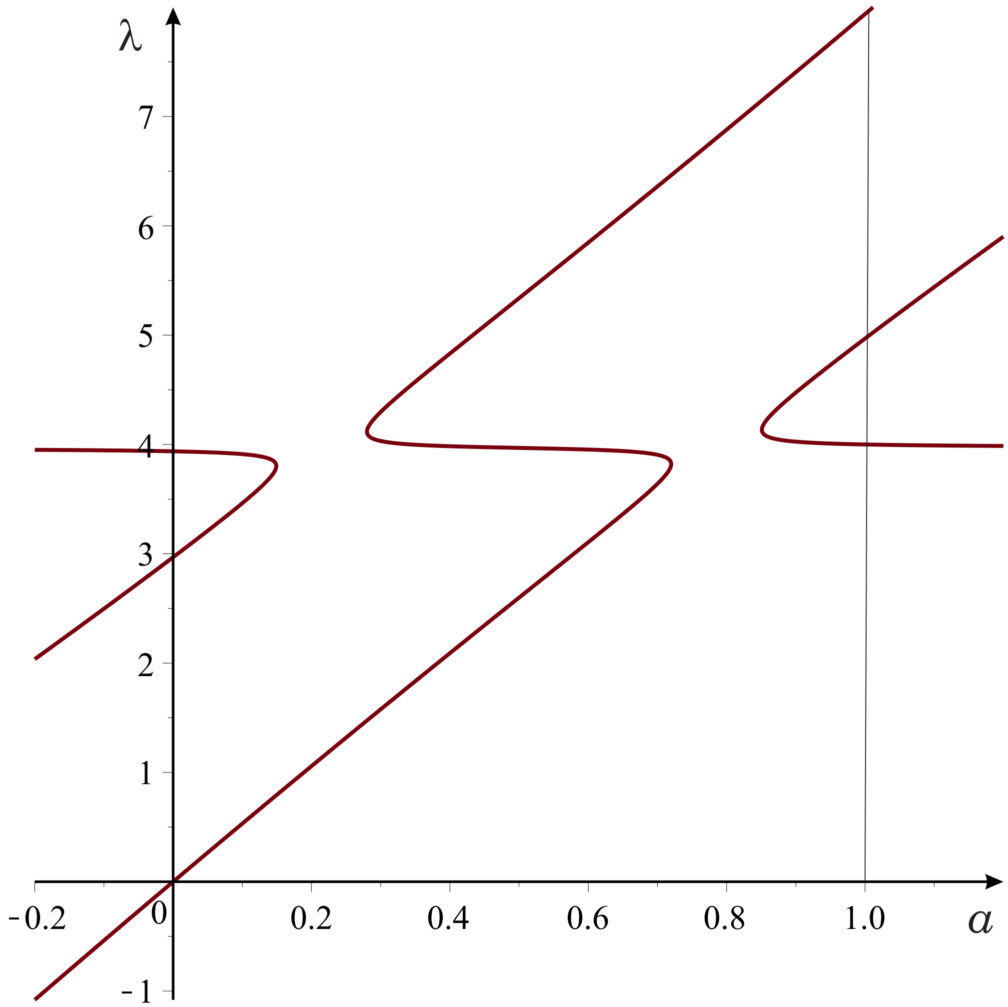

Fig. 11. $\alpha_1 = \alpha_2 = 3,\ \alpha_0 = \alpha_3 = 63/32$

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