Quasi-exceptional domains

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Abstract

Exceptional domains are domains on which there exists a positive harmonic function, zero on the boundary and such that the normal derivative on the boundary is constant. Recent results classify (under some mild additional assumptions) exceptional domains as belonging to either a certain one-parameter family of simply periodic domains or one of its scaling limits.

We introduce quasi-exceptional domains by allowing the boundary values to be different constants on each boundary component. This relaxed definition retains the interesting property of being an *arclength quadrature domain*, and also preserves the connection to the hollow vortex problem in fluid dynamics. We give a partial classification of such domains in terms of certain Abelian differentials. We also provide a new two-parameter family of periodic quasi-exceptional domains. These examples generalize the hollow vortex array found by Baker, Saffman, and Sheffield (1976). A degeneration of regions of this family provides doubly-connected examples.

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1. Introduction

A domain $D \in \mathbb{R}^n$ is called exceptional if there is a positive function u (called a *roof function*) harmonic in D, zero on the boundary, and

$$\frac{\partial}{\partial n} u(z) = 1, \quad z \in \partial D, \tag{1}$$

where the differentiation is along the normal pointing inwards into D, and it is assumed that the boundary is smooth. Evident examples are exteriors of balls and half-spaces. For $n > 2$ the only other known examples are cylinders whose base is an exceptional domain in $\mathbb{R}^2$. If the smoothness assumption on the boundary is dropped, then there are also certain cones in higher dimensions and pathological "non-Smirnov" examples in the plane [9].

The problem of description of all exceptional domains in the plane was stated in [8] and settled in [9] under a topological assumption which was removed in [16] using an unexpected correspondence to minimal surfaces. The first non-trivial example was given in [8]. This example appeared in another context related to fluid dynamics in [12]. A second non-trivial example was noticed in [9] and [16]. This example had also appeared previously in studies of fluid dynamics [2] (see also [5]).

Let us introduce *quasi-exceptional* domains, by relaxing the definition to allow the Dirichlet condition to be a different constant on each boundary component. Thus, a domain $D \in \mathbb{R}^n$ is called quasi-exceptional if there is a positive harmonic function u in D, which is constant on each boundary component (but not necessarily the same constant) and the Neumann condition (1) holds. We will continue to call u a *roof function*. Again, we assume that each component of the boundary is smooth.

Added in press: In the interesting preprint [17], Martin Traizet has considered an even more general problem, allowing the Neumann data to take different signs on different components. As with the current paper, [17] is motivated by the hollow vortex problem from fluid dyanmics. Extending [16], a correspondence to minimal surfaces is given in [17], and techniques from minimal surface theory are used to produce new examples.

We summarize several interesting aspects of exceptional domains. These statements all hold true for quasi-exceptional domains.

- Fluid dynamics: As noted above, the two non-trivial examples first appeared in fluid dynamics [12, 2]. In general, one can interpret exceptional domains in terms of a *hollow vortex* problem. The level lines of u can be interpreted as stream lines of a two-dimensional stationary flow of ideal fluid, and condition (1) expresses the fact that the pressure is constant on the boundary. Such conditions may exist if the components of the complement of D are air bubbles in the surrounding liquid. Notice that the rotation of the fluid around all bubbles corresponding

to exceptional domains is in the same direction because $\partial u/\partial n > 0$ on the boundary.

- Quadrature domains [6]: Exceptional domains provide examples of arclength null-quadrature domains, that is, domains for which integrals over ∂D with respect to arc length of every analytic function in the Smirnov class $E^1(D)$ vanishes.

- Differentials on Riemann surfaces: By way of the connection to quadrature domains, the study [6] indicates a connection to half-order differentials. We make use of Abelian differentials in Section 4 below.

- The Schwarz function of a curve: In [9], it was noticed that the function $u(z)$ satisfies
$$\partial_z u(z) = \sqrt{-S'(z)},$$
where $S(z)$ is the Schwarz function of $\partial\Omega$ and $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ is the Cauchy-Riemann operator.

- Minimal surfaces: the work [16] established a nontrivial correspondence between exceptional domains and a special type of minimal surfaces called a "minimal bigraphs". This correspondence has been extended to quasi-exceptional domains in [17], but the minimal surfaces in that case need not be embedded. This prevents applying the results on complete embedded minimal surfaces that were used in [16] to classify exceptional domains.

The classification results for exceptional domains of finite connectivity show that they are quite restricted; all examples can be conformally mapped onto a disk by elementary functions.

Problem 1: Classify quasi-exceptional domains.

We begin to address this problem below, give a partial classification of periodic and finitely-connected exceptional domains, and provide new periodic and doubly-connected examples described in terms of elliptic functions. First, we explain the relation to arclength null-quadrature domains.

2. Arclength null-quadrature domains

A bounded domain $D \subset \mathbb{C}$ is a *quadrature domain* if it admits a formula expressing the area integral of every function f analytic and integrable in D as

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a finite sum of weighted point evaluations of the function and its derivatives. i.e.

$$\int_D g(z)dA(z) = \sum_{m=1}^{N} \sum_{k=0}^{n_m} a_{m,k} g^{(k)}(z_m), \qquad (2)$$

where z_m are distinct points in D and $a_{m,k}$ are constants independent of g.

A (necessarily unbounded) domain $D \subset \mathbb{C}$ is called a *null-quadrature domain* (NQD) if the area integral of every function g analytic and integrable in D vanishes:

$$\int_D g(z)dA(z) = 0. \qquad (3)$$

M. Sakai [15] completely classified NQDs in the plane.

Following [9] we will refer to a domain $D \subset \mathbb{C}$ as an *arclength null-quadrature domain* (ALNQD) if the integral over ∂D of every function g in the Smirnov class $E^1(D)$ vanishes (in the case ∞ is an isolated point on ∂D, we take the restricted class of functions $g(z) \in E^1(D)$ vanishing at infinity):

$$\int_{\partial D} g(z)ds(z) = 0. \qquad (4)$$

The Smirnov class $E^1(D)$ is not the same as the Hardy space $H^1(D)$. Namely, a function g analytic in D is said to belong to $E^1(D)$ if there exists a sequence of domains $D_1 \subset D_2 \subset \ldots$ with $\cup_k D_k = D$ with rectifiable boundaries such that:

$$\sup_k \int_{\partial D_k} |g(z)||dz| < \infty.$$

One may also define quadrature domains in higher dimensions using a test class of harmonic functions, but we will restrict ourselves to the case of $n = 2$ dimensions.

Inspired by the successful classification of NQDs [15], the problem of classifying ALNQDs was suggested in [9]. We pose this problem again while stressing that it does not reduce to the classification of exceptional domains (whereas it might reduce to classification of *quasi-exceptional* domains).

Problem 2: Classify ALNQDs.

The following proposition shows that quasi-exceptional domains are ALNQDs. Thus, the new examples (described in the last section) of quasi-exceptional domains also provide new ALNQDs. Problem 2 is closely related

to Problem 1, and if the converse of the proposition is true then the two problems are equivalent.

Proposition 1. *Suppose that D is a quasi-exceptional domain. Then D is an ALNQD.*

Proof. Consider the complex analytic function $F(z) = u_x - iu_y$, where u is the roof function. We will need the following Claim which is proved in the next section (see Lemma 2).

Claim. *The roof function u of D satisfies $\nabla u(z) = O(1)$ in D. Thus, $F(z)$ is bounded.*

Suppose that g is in the Smirnov space $E^1(D)$. Using the fact that $ds = iF(z)dz$, we have

$$\int_{\partial D} g(z)ds = \int_{\partial D} ig(z)F(z)dz. \tag{5}$$

As F is bounded, $gF \in E^1(D)$. If ∞ is not an isolated boundary point, the integral equals zero by Cauchy's theorem.

If ∞ is an isolated boundary point, then we have $u(z) = \log|z| + \text{const} + O(1/z)$, so $F(z) = O(1/z)$, $z \to \infty$. Now $g(\infty) = 0$, so Fg has a zero of order at least 2 at ∞, and the integral is zero again.

3. A potential theoretic restriction on the roof function

We restrict ourselves to the case $n = 2$, and assume that the order of connectivity of D is finite, or that the roof function u is periodic, and a fundamental region for D has finite connectivity.

Recall that a Martin function is a positive harmonic function M in a domain Ω with the property that for any positive harmonic function v in Ω, the condition $v \leq M$ implies that $v = cM$, where $c > 0$ is a constant. (Often, Martin functions are called minimal harmonic functions - cf. [7].) Martin functions on finitely connected domains are simply Poisson kernels evaluated at points of the Martin boundary, the boundary under Caratheodory compactification (prime ends) of the domain (see [3]).

Any domain D of finite connectivity in $\mathbb{C}$ is conformally equivalent to a circular domain Ω. A circular domain is a domain whose boundary components are points or circles. For a circular domain, a Martin function M can be of two types:

a) There is a component of $\partial\Omega$ which is a single point z_0, and M is proportional to the Green function of $\Omega \cup \{z_0\}$ with the singularity at z_0.

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b) There is a point $z_0 \in \partial\Omega$ which is not a component of $\partial\Omega$, and M has zero boundary values at all points of $\partial\Omega\backslash\{z_0\}$. The local behavior in this case is like $-\text{Im}\,(1/z)$ in the upper half-plane near 0.

Let D be an exceptional domain, and u a harmonic function with the property (1). The following result was proved for exceptional domains by the current first author, but communicated in [9, Thm. 4.2]. Here we repeat the proof with minor adjustments.

Lemma 2. *The roof function u of a quasi-exceptional domain satisfies $\nabla u(z) = O(1)$ in D. Moreover, u is a sum of a bounded harmonic function and at most two Martin functions.*

Proof. We follow the second part of the proof from [9, Thm. 4.2].

Let $R > 0$ and consider an auxiliary function

$$w_R = \frac{|\nabla u|}{u + R},$$

where $R > 0$ is a parameter. A direct computation shows that

$$\Delta \log w_R = w_R^2, \tag{6}$$

and $w_R(z) = 1/(c_k + R) \leq 1/R$ for $z \in \partial D$, where $c_k \geq 0$ are the constants taken in the Dirichlet condition. We claim that

$$w_R(z) \leq 2/R, \quad z \in D, \tag{7}$$

from which the result follows by letting $R \to \infty$ which gives $|\nabla u| \leq 2$ in D.

Suppose, contrary to (7), that $w_R(z_0) > 2/R$, for some $z_0 \in D$. Let

$$v(z) = \frac{2R}{R^2 - |z - z_0|^2}, \quad z \in B(z_0, R) = \{z : |z - z_0| < R\}.$$

Obviously, $v(z) \geq 2/R$. A computation reveals that $\Delta \log v = v^2$. Let

$$K = \{z \in D \cap B(z_0, R) : w_R(z) > v(z)\}.$$

We have $z_0 \in K$, since $v(z_0) = 2/R$. Let K_0 be the component of K, containing z_0. Then we have $w_R(z) = v(z)$ on ∂K_0, since $w_R(z) < v(z)$ on $\partial D \cap B(z_0, R)$ while $v(z) = +\infty$ on $\partial B(z_0, R)$. On the other hand,

$$\Delta(\log w_R - \log v) = w_R^2 - v^2 > 0 \quad \text{in} \quad K_0.$$

So the subharmonic function $\log w_R - \log v$ is positive in K_0 and vanishes on the boundary, a contradiction.

This proves that $\nabla u = O(1)$. In order to see the second statement, we note that $\nabla u = O(1)$ implies that $u(z) = O(|z|)$ has order one. The result then follows by first solving the Dirichlet problem (with a bounded function) having the same boundary values as u; subtracting this function, one may then apply [10, Theorem II].

4. Partial classification in terms of Abelian differentials

Let D be a QE domain of one of the following types:

Type I. D is finitely connected, or

Type II. D/Γ is finitely connected, where Γ is the group of transformations $z \mapsto z + n\omega$, and $u(z + \omega) = u(z)$ for some $\omega \in \mathbb{C}\backslash\{0\}$. We call this the periodic case. (As above, u is the roof function.)

In this section we give a classification of QE domains of these two types in terms of Abelian differentials of a compact Riemann surface with an anti-conformal involution.

If D is of type I, and ∞ is an isolated boundary point, then $D' = D\cup\{\infty\}$ is conformally equivalent to some bounded circular domain Ω, and we suppose that $p \in \Omega$ corresponds to ∞. If ∞ is not isolated, we put $D' = D$, and Ω is a bounded circular domain conformally equivalent to D'. In any case, we have a conformal map $\phi : \Omega \to D'$, which may have at most one simple pole at $p \in \Omega$.

If D is of type II, let $G = D/\Gamma$. The Riemann surface G is a finitely connected domain on the cylinder $\mathbb{C}/\Gamma$; this cylinder is conformally equivalent to the punctured plane, and we identify it with $\mathbb{C}^*$. Then $G \subset \mathbb{C}^*$ must have one or two punctures of $\mathbb{C}/\Gamma$ as isolated boundary points, and we denote by G' the union of G with these isolated boundary points. Then G' is conformally equivalent to a bounded circular domain of finite connectivity Ω in which there is one or two points a and b correspond to the added punctures. We have a multi-valued conformal map $\phi : \Omega \to D$.

The points a and b are logarithmic singularities of ϕ.

We pull back u on Ω : set $v = u \circ \phi$. As u is periodic, v is a single-valued positive harmonic function on $\Omega\backslash\{a, b\}$.

Consider the differential on Ω

$$dv = v_z dz = (1/2)(v_x - iv_y)(dx + idy) = g(z)dz.$$

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This is well defined on Ω: g is a single-valued meromorphic function in Ω with at most simple poles at p or a and b. Indeed, for a positive harmonic function, an isolated singularity is either removable or logarithmic. In the second case the gradient has a simple pole.

Next, we extend v as a multi-valued function to a compact Riemann surface S. Let Ω' be the mirror image of Ω; we glue it to Ω in the standard way (along each circular boundary component) and obtain a compact Riemann surface S. We denote by $\sigma : z \mapsto z^*$ the anti-conformal involution which fixes the boundary components of Ω. The Riemann surface S is of genus g, and the involution σ has fixed set corresponding to $\partial\Omega$, which consists of $n = g + 1$ ovals. Such involutions are called involutions of maximal type.

Each branch of v is constant on each boundary component, so it extends through this boundary component by reflection to the double S of Ω. The extensions of various branches of v through different boundary components do not match: they differ by additive constants. On the other hand, the differential dv is well defined on the double. Namely,

$$(dv)^* = -dv, \tag{8}$$

where $*$ is the action of involution on differentials. Thus we have a meromorphic differential dv on S.

Choose a basis of 1-homology in S so that the A-loops are simple closed curves in Ω, each homotopic to one boundary component of Ω, and the B loops are dual to the A-loops. For Type I, all periods over A-loops are purely imaginary, because

$$v = \mathrm{Re} \int dv$$

is single-valued. For Type II these periods are imaginary except those which correspond to simple loops around one pole, a or b.

Now we discuss ϕ, or better the differential $d\phi = \phi'(z)dz$. We have, from the condition that our domain is quasi-exceptional:

$$2|dv| = |d\phi| \quad \text{on} \quad \partial\Omega.$$

The ratio of two differentials is a function. So we have a meromorphic function B on Ω such that

$$2Bdv = d\phi. \tag{9}$$

This function has absolute value 1 on $\partial\Omega$. Therefore, it extends to S by symmetry. It has no zeros in Ω because $d\phi$ has no zeros. Its poles in Ω must

match the zeros of dv, because $d\phi$ is zero-free (indeed, ϕ is univalent). In fact, B is a meromorphic function on S. To justify this claim when dv has a singularity on $\partial\Omega$, we observe that this singularity is removable for B as follows from the next lemma.

Lemma 3. *Consider the equation*

$$\phi' = Bh,$$

where h is meromorphic in a neighborhood V of 0, B is holomorphic and zero-free in $V\backslash\{0\}$, $|B(z)| = 1$ for $z \in V \cap \mathbb{R} \backslash \{0\}$, and ϕ is univalent in $\{z \in V : \operatorname{Im} z > 0\}$. Then the singularity of B at 0 is removable.

Before proving the lemma, we note that in order to apply it in our setting we compose B with a linear fractional transformation that sends V to a neighborhood of the singularity we wish to remove such that the real line is mapped to the circular boundary component with 0 sent to the singularity.

Proof. Proving this by contradiction, assume that 0 is an essential singularity of B. By symmetry we have $B(\overline{z}) = 1/\overline{B(z)}$. We claim that there exists a sequence $z_k \to 0$ such that

$$\liminf_{k\to\infty} |z_k \log |B(z_k)|| > 0. \tag{10}$$

Indeed, suppose that this is not so. Then $\log|B(z)| = o(z^{-1})$, and the Phragmén–Lindelöf theorem, see, for example, [11, Ch. I, Thm. 22], implies that B has a limit as $z \to 0$. By choosing a subsequence and using the symmetry we can find a sequence in the upper half-plane with the property

$$\liminf_{k\to\infty} |z_k| \log |B(z_k)| > 0. \tag{11}$$

or with the property

$$\liminf_{k\to\infty} |z_k| \log |B(z_k)| < 0. \tag{12}$$

Distortion theorems for univalent functions imply that

$$c(\operatorname{Im} z)^3 \le |\phi'(z)| \le C(\operatorname{Im} z)^{-3}, \tag{13}$$

In addition to this, we have for some integer m

$$c|x|^m \le |\phi'(x)| \le C|x|^m, \quad z \in V \cap \mathbb{R}, \tag{14}$$

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because h is meromorphic and $|B(x)| = 1$ for $z \in V \cap \mathbb{R}$. Taking $n = \max\{3, -m\}$, we obtain that subharmonic function $u(z) = \log^+ |z^n \phi'(z)/C|$ satisfies $u(x) = 0$, $x \in V \cap \mathbb{R}$, and $u(re^{i\theta}) \leq \psi(\theta)$, where $\psi(\theta) = -3 \log \sin \theta$, $\theta \in (0, \pi)$. As

$$\int_0^\pi \psi(\theta) d\theta < \infty,$$

we can apply Carleman's "loglog" theorem [4], [14], and conclude that u is bounded from above in the intersection of V with the upper half-plane. This contradicts (11). If (12) holds, one applies the same argument to $1/\phi'$. This completes the proof of the lemma.

We can thus restate the problem of finding QE domains (under the restrictions we impose) as follows:

Proposition 4. *All QE domains of the Types I and II are parametrized by triples $(S, d\omega, B)$, where S is a compact Riemann surface with an involution of "maximal type" (the complement of the fixed set of the involution consists of two regions homeomorphic to plane regions), $d\omega$ is a meromorphic differential that enjoys the symmetry property (8), and B is the function which has the symmetry property*

$$B^*(z) := \overline{B(z^*)} = 1/B(z),$$

and has poles at the zeros of $d\omega$ on one half of S, that is in Ω. There is an additional condition that

$$\phi = 2 \int B d\omega \tag{15}$$

is globally univalent and single-valued in case I, and single-valued except the residues in case II.

To recover D from a triple (S, dw, B) one takes one of the components $\Omega \subset S$ complementary to the fixed set of the involution. Then $D = \phi(\Omega)$, where ϕ is defined in (15).

In order to check the condition on the global univalence of ϕ, it is sufficient to verify that periods of $d\omega/B$ are zero on the boundary curves, and that these boundary curves are mapped by ϕ injectively.

A general conclusion is the following.

Proposition 5. *The boundary of a quasi-exceptional domain of type I or II is parametrized by an Abelian integral.*

Next we provide a partial classification of quasi-exceptional domains in terms of the data stated in the above formulation.

Theorem 6. *The differential dv has either two or four poles on S, counting multiplicity. Moreover, if dv has two poles in S then D is either a complement of a disk or a half-plane.*

Remark. If $B \neq \text{const}$, then $1/B$ is an Ahlfors function of Ω.

Proof. Let us first show that dv has some poles. Otherwise, dv is holomorphic, and thus u is bounded. Let $z_1 \in \partial D$ and $z_2 \in \partial D$ be the points where u assumes its maximal and minimal values. Then du/dn has opposite signs at these two points which contradicts (1).

The differential dv has at most simple poles at p, a, and b (whichever of these points are present) and at their images σp, σa, and σb. In addition it may have double poles on $\partial \Omega$. The total number of poles (without multiplicity) in $\overline{\Omega}$ is at most two by Lemma 2. Thus on S, the differential dv has two or four poles, counting multiplicity.

Notice that v is constant on each boundary component, so the gradient is perpendicular to the boundary $\partial \Omega$, so the total rotation of this gradient, as we describe the boundary is the same as the total rotation of the tangent vector to the boundary. This is equal to $2\pi(2-n)$ because the outer boundary component is traversed counterclockwise and the rest clockwise, as parts of the boundary of Ω. So v_z which is conjugate to the gradient, rotates $n-2$ times.

From this we can conclude how many zeros dv has in Ω. The number N of zeros of dv in Ω satisfies

$$n - 2 = N - (\text{the number of poles in } \Omega), \tag{16}$$

where a double pole on $\partial \Omega$ is counted as a single pole in Ω. This formula is well-known.

Suppose that dv has exactly 2 poles, counting multiplicity. This can occur in one of three ways:

(1) dv has a simple pole at p in Ω.

(2) dv has one double pole at $z_0 \in \partial \Omega$.

(3) dv has a simple pole at a in Ω (and b does not exist).

If Case (1) holds, then ∞ is an isolated point on ∂D, and by Proposition 1, D is an arclength quadrature domain with quadrature point at ∞. It now follows from [6, Remark 6.1] that D is the exterior of a disk.

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In Case (2), we will show that B is constant. First note that $d\phi$ has a double pole at z_0, so B does not have a zero or a pole at z_0. Since ϕ is a conformal map, it follows from (9) that B has no zeros and N poles in Ω (located at the zeros of dv). Assume for the sake of contradiction that B is not constant. By Lemma 3, B is meromorphic in S, and by Lemma 2, $1/|B|$ is bounded by a constant in Ω. Since $|B| = 1$ on $\partial\Omega$, B thus maps Ω to the exterior of the unit disk and maps each of the n components of $\partial\Omega$ to the unit circle. This implies that B has at least n poles in Ω. Combined with (16), this gives the contradiction $N = n - 1 \geq n$. We conclude that B is constant which implies that the gradient of the roof function is constant. Thus, the roof function is linear, and D is a halfplane.

In Case (3), the behavior of ϕ at point a is logarithmic, so $d\phi$ has a simple pole at a and B does not have a zero or a pole at a. Arguing as before, we conclude that B is constant and that D is a halfplane.

Corollary 7. *The only QE domains with compact boundary are exteriors of disks, and the only QE domains of types I, II with one unbounded boundary component are half-planes.*

If D is a quasi-exceptional domain that is not a disk or halfplane, then dv has four poles, and more precisely, we have the following two possibilities.

D is of *type I*: dv has two double poles on $\partial\Omega$. This implies that the boundary ∂D consists of two simple curves tending to ∞ in both directions, and $n - 1$ bounded components. The unbounded components are the ϕ-images of two arcs of one boundary circle of Ω which contains both singularities of ϕ and v.

D is of *type II*: dv has two simple poles in Ω. In this case D must be periodic, all components of ∂D are compact and there are n such components per period.

Note that the possibility that dv has one simple pole in Ω and one double pole on $\partial\Omega$ is excluded by Lemma 2: it is easy to see that in this case the number of Martin functions in the decomposition of u would be infinite.

We have thus described possible topologies of the QE domains satisfying the assumptions stated in the beginning of this section.

In the next section we construct the examples of types I and II with S of genus 1. We conjecture that there exist QE domains of types I and II based on S having any genus.

5. New examples

Description of our examples requires elliptic functions (all known exceptional domains can be parametrized by elementary functions).

Example of type I.

Let G be the rectangle with vertices $(0, 2\omega_1, 2\omega_1 + \omega_3, \omega_3)$, where $\omega_1 = 2\omega$, $\omega > 0$, and $\omega_3 = \omega'$, where $\omega' \in i\mathbb{R}$, $\omega'/i > \omega$. Let G' be the reflection of G in the real line. The union of G, G' and the interval $(0, 2\omega_1)$ make a fundamental domain of the lattice Λ generated by $2\omega_1, 2\omega_3$.

Let us consider the ω_1-periodic positive harmonic function h in G which is zero on the horizontal segments of the boundary ∂G, except for one singularity per period, at 0, where it behaves in the following way:

$$h(z) \sim -\mathrm{Im}\,(1/z), \quad z \to 0.$$

Note that the existence of h is clear as it can be expressed (through conformal mapping) in terms of the Poisson kernel of a ring domain.

Function h has two critical points in G, at w_1 and w_2 with $\mathrm{Re}\, w_1 = \omega_1/2$ and $\mathrm{Re}\, w_2 = 3\omega_1/2$, while the imaginary parts of w_1, w_2 are equal. Let us choose real constants c_1 and c_2 such that $v = 2(h + c_1 y) + c_2$ is a positive harmonic function with critical points $\omega_1/2 + \omega_3/2$ and $3\omega_1/2 + \omega_3/2$. The existence of such constants c_1 and c_2 is evident by continuity.

The z-derivative $\partial_z v = (v_x - iv_y)/2$ is an elliptic function with periods $\omega_1, 2\omega_3$, and thus also elliptic with periods Λ. Asymptotics near 0 show that $\partial_z v \sim -i/z^2$, and as this function has only one pole per period, (with respect to the parallelogram $\omega_1, 2\omega_3$), we have $\partial_z v = -i\wp + ic_0$, where $\wp$ is the Weierstrass function corresponding to the lattice $(\omega_1, 2\omega_3)$.

Zeros of $\partial_z v$ in $G \cup G'$ are $\omega_1/2 + \omega_3/2, 3\omega_1/2 + \omega_3/2$ and complex conjugates in G'.

Let B be an elliptic function with periods $2\omega_1, 2\omega_3$ having simple poles at $\omega_1/2 + \omega_3/2, 3\omega_1/2 + \omega_3/2$, and zeros at complex conjugate points. Such function exists by Abel's theorem: the sum of zeros minus the sum of poles equals $-2\omega_3$. This function is unique up to a constant factor. By symmetry, $B(\overline{z}) = c/\overline{B(z)}$, so on the real line $|B(x)|^2 = c$ and we can choose the constant factor in the definition of B so that $c = 1$. Thus

$$|B(x)| = 1, \quad x \in \mathbb{R}. \tag{17}$$

Then we have $B(x + \omega_3)\overline{B(x - \omega_3)} = 1$, but by periodicity we also have $B(x + \omega_3) = B(x - \omega_3)$, thus $|B(x + \omega_3)| = 1$. So

$$|B(z)| = 1 \quad \text{on the horizontal segments of} \quad \partial G. \tag{18}$$

Now we consider the function

$$F = \frac{\partial v}{\partial z} B = (-i\wp + ic_0)B.$$

This function F is holomorphic and zero-free in G (the zeros of $\partial v/\partial z$ in G are exactly canceled by the poles of B). Let us show that

$$\int_0^{2\omega_1} F(x + iy)dx = 0, \quad y \in (0, \omega_3). \tag{19}$$

This property follows from the fact that $B(z)$ and $B(z + \omega_1)$ have the same poles but the residues at these poles are of the opposite signs, because B has only two poles in the period parallelogram. Thus

$$B(z + \omega_1) = -B(z). \tag{20}$$

Property (20) and ω_1-periodicity of $\wp$ imply (19).

As F has no zeros, the primitive $f = \int F$ is locally univalent. Assuming for the moment that it is univalent, it maps G onto some region in the plane, and we have

$$|f'| = |F| = \left|\frac{\partial v}{\partial z}\right||B|.$$

Define u by composing v with f^{-1}, so $u(f(z)) = v(z)$. Then u is positive and harmonic in $f(G)$. Taking into account (18), we conclude that u satisfies (1) $f(G)$ is a quasi-exceptional domain. Note that, in accordance with the previous results in [16], $f(G)$ is not an exceptional domain since the piecewise constant Dirichlet data is not the same constant on each boundary component.

In order to show that f is in fact univalent, it is enough to show that it is one-to-one on the the horizontal sides of G (since f is locally univalent). To this end, we make the following claims:

Claim 1: $\operatorname{Re} f$ is increasing along the segment $[\omega', \omega' + 2\omega]$ and decreasing along the segment $[\omega' + 2\omega, \omega' + 4\omega]$.

Claim 2: $\operatorname{Im} f < \operatorname{Im} f(\omega')$ on $(\omega', \omega' + 2\omega]$ and $\operatorname{Im} f > \operatorname{Im} f(\omega')$ on $[\omega' + 2\omega, \omega' + 4\omega)$.

Claim 3: $\operatorname{Im} f$ achieves its minimum and maximum values along $[\omega', \omega' + 4\omega]$ at $\omega' + \omega$ and $\omega' + 3\omega$ respectively.
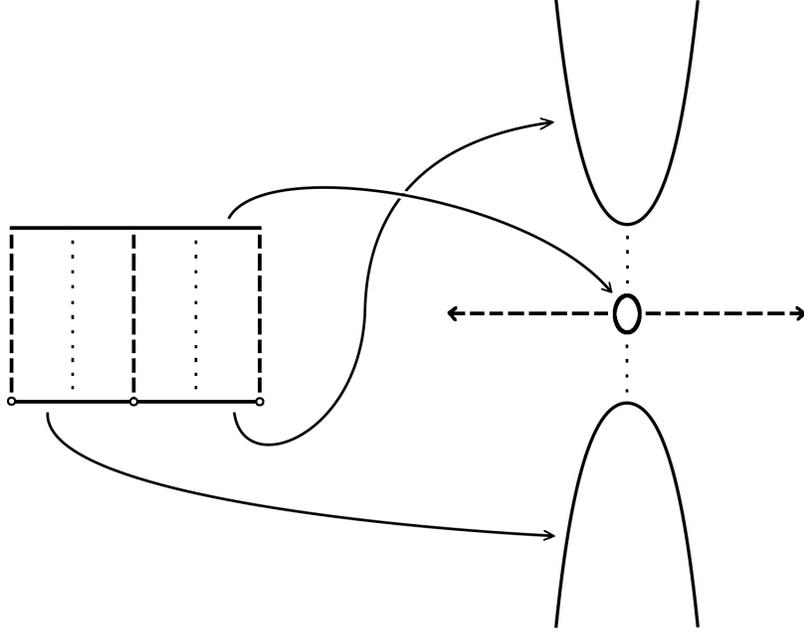
Figure 1: A doubly-connected quasi-exceptional domain of type I mapped from the rectangle G.

Claim 4: $\operatorname{Re} f$ is increasing on the segment $[0, 2\omega]$, and $\operatorname{Re} f$ is decreasing along $[2\omega, 4\omega]$.

Claim 5: $\operatorname{Im} f$ attains its maximum along $[0, 2\omega]$ at ω and its minimum along $[2\omega, 4\omega]$ at 3ω.

Claim 6: $\operatorname{Im} f(\omega) < \operatorname{Im} f(\omega' + \omega) < \operatorname{Im} f(\omega' + 3\omega) < \operatorname{Im} f(3\omega)$.

Claim 1 implies that $\operatorname{Re} f$ is monotone along each of the named segments, and since $\operatorname{Im} f$ differs between the two segments by Claim 2, f must be one-to-one on the top side of G. Claim 4 implies that f is one-to-one on each of the two segments on the bottom side of G. Claims 3, 5, and 6 imply that the images of these three segments do not intersect each other. This shows that f is one-to-one on the horizontal sides of G.

The claims can be established by the properties of $f' = F = \partial_z v B$. First note that, since $v(z)$ is positive in G and vanishes on the horizontal sides of G, we have $\partial_x v(z) = 0$ on both sides, and for $x \in \mathbb{R}$ we have $\partial_y v(x + \omega_3) < 0$, and $\partial_y v(x) > 0$. In particular, $i\partial_z v(z) = i(\partial_x v - i\partial_y v)/2 = \partial_y v/2$ is real. The function $B(z)$ is a Jacobi sn function, whose properties are well-known [1,

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Section 47]. $B(z)$ sends the top side of G to the unit circle, such that the four segments $[\omega', \omega'+\omega]$, $[\omega'+\omega, \omega'+2\omega]$, $[\omega'+2\omega, \omega'+3\omega]$, and $[\omega'+3\omega, \omega'+4\omega]$ correspond to the fourth, third, second, and first quadrants of the unit circle, respectively. Multiplication by $\partial_z v(z)$ distorts this circle and rotates it by an angle of $\pi/2$ (since $\partial_z v(z)/i$ is positive) but preserves the two-fold symmetry. This determines the sign of the real and imaginary parts of f'. Since $dz = dx$ is purely real on the horizontal sides of G, this gives the monotonicity of $\operatorname{Re} f$ stated in Claim 1. Claims 2 and 3 follow from the sign of $\operatorname{Im} f'$ and the fact that $\operatorname{Im} f'$ is an odd function with respect to reflection in each of the points $\omega' + \omega$ and $\omega' + 3\omega$.

The four segments $[0, \omega]$, $[\omega, 2\omega]$, $[2\omega, 3\omega]$, and $[3\omega, 4\omega]$ on the bottom side of G are sent to the second, first, fourth, and third quadrants of the unit circle respectively. Since $\partial_z v(z)/i$ is negative along the bottom side of G, under $f'(z)$ this becomes the first, fourth, third, and second quadrants, respectively. This establishes Claim 4, and combined with the reflectional symmetry, also Claim 5. Claim 6 follows from the fact that $\partial_z v(z)B(z) > 0$ along the vertical segment $[\omega, \omega+\omega']$ and $\partial_z v(z)B(z) < 0$ along $[3\omega, 3\omega+\omega']$.

Remark. For the purpose of plotting Figure 1, instead of the above construction, we expressed F as a ratio of Weierstrass σ functions:

$$f'(z) = F(z) = \frac{\sigma(z - \omega + \omega'/2)^2 \cdot \sigma(z - 3\omega + \omega'/2)^2}{\sigma(z)^2 \cdot \sigma(z - 2\omega) \cdot \sigma(z - 6\omega + 2\omega')},$$

where σ is a Weierstrass σ function with fundamental "periods" 4ω, $2\omega'$ (but recall that σ is not itself periodic). As usual, the shifts are chosen based on the the zeros and poles of F, but one of the shifts must be replaced by an equivalent lattice point in a different rectangle in order to satisfy [1, Eq. (1), Sec. 14]. This explains why one of the poles is placed at $6\omega - 2\omega'$.

Example of type II.

Only small modifications of the previous example are needed. Using the same G, G', ω_1, ω_3, we define h as the ω_1-periodic function, positive and harmonic in G' except two logarithmic poles at $i\epsilon$ and $\omega_1 + i\epsilon$, where $\epsilon \in (0, \omega_3/2)$. Then we can find constants c_1 and c_2 such that $v = h + c_1 y + c_2$ has critical points at $\omega_1/2 + \omega_3/2$ and $3\omega_1/2 + \omega_3/2$.

Then v_z is an elliptic function with periods ω_1, $2\omega_3$ with two simple poles at $i\epsilon$ and $-i\epsilon$ per period parallelogram. This elliptic function has the form

$$\frac{-i\wp}{1 + c\wp} + ic_0$$

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with some small real c. The rest of the construction is the same as in the previous example.

In a similar manner to the above, in order to plot the figures, we expressed F as a ratio of Weierstrass σ functions:

$$f'(z) = F(z) = \frac{\sigma(z - \omega + \omega'/2)^2 \cdot \sigma(z - 3\omega + \omega'/2)^2}{\sigma(z - i\epsilon) \cdot \sigma(z + i\epsilon) \cdot \sigma(z - 2\omega - i\epsilon) \cdot \sigma(z - 6\omega + i\epsilon + 2\omega')}.$$

Figure 2: An example of Type II with $\omega_1 = 2$, $\omega_3 = 2$ and $\epsilon = 0.5$. Note that we have aligned the array horizontally in order to plot two periods.

Figure 3: An example of Type II with $\omega_1 = 2$, $\omega_3 = 1.5$, and $\epsilon = 0.4$. Note that we have aligned the array horizontally in order to plot two periods.

6. Hollow vortex equilibria

Let G_j be smooth Jordan domains on the plane whose closures are disjoint, and

$$D = \mathbb{C} \setminus \cup_j D_j.$$

Let F be a complex potential of a flow of an ideal fluid which is divergence-free and locally irrotational in D. If the pressure (determined by $|F'|$ according

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to Bernoulli's law) is constant on ∂D then G_j can be interpreted as constant-pressure gas bubbles in the flow.

The first examples of this situation, with two bubbles were constructed by Pocklington [13]. Periodic exceptional domains give periodic examples with one bubble per period, with the flow on the surface on the bubbles rotating in the same direction [2] (see also [5]). Crowdy and Green [5] constructed periodic examples with two bubbles per period rotating in the opposite direction. Our example of type II can be interpreted as a periodic flow with two bubbles per period rotating in the same direction.

The velocity at infinity in our examples is directed in the opposite directions on the two sides of the row of the bubbles.

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