Spherical metrics with conic singularities

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Let S be a compact surface of genus g, $A = (a_1, \ldots, a_n)$ a finite sequence of points S, and $\alpha = (\alpha_1, \ldots, \alpha_n)$ positive numbers. Consider the set of Riemannian metrics $\mathrm{Sph}_{g,\alpha}$ of constant curvature 1 on $S \backslash A$, with conic singularities at a_j with angles $2\pi\alpha_j$.

There is an explicit necessary and sufficient condition for the set $\mathrm{Sph}_{g,\alpha}$ to be non-empty, in terms of g, α, see [3], [4], [1].

Since every such metric defines a conformal structure, we have the *forgetful map* $\pi : \mathrm{Sph}_{g,\alpha} \to \mathrm{Mod}_{g,n}$, to the moduli space of compact Riemann surfaces of genus g with n punctures.

The main problem is what the valence of this forgetful map can be.

It can be infinite for the followings simple reason. Suppose that a metric in $\mathrm{Sph}_{g,n}$ is given. Consider the *developing map* from the universal cover to the sphere equipped with the spherical metric. $f : \widetilde{S \backslash A} \to \overline{mathbf C}$. The monodromy representation corresponding to this map consists of rotations of the sphere. And conversely, if we have such a map whose monodromy consists of rotations of the sphere, and with given asymptotic behaviors at the points a_j reflecting the conic singularities at these points, one can recover the metric by pulling back the metric from the sphere. Two metrics are the same if their developing maps satisfy $f_1 = \phi \circ f_2$ for some rotation ϕ of the sphere. But it may happen that two developing maps are related by

$$f_1 = \psi \circ f_2, \tag{1}$$

where ψ is a linear fractional transformation which is not a rotation. (This happens in the exceptional case when the monodromy is conjugate to the subgroup of the unit circle, so called co-axial monodromy).

Let us call two metrics *equivalent* if their developing maps are related by (1).

Conjecture. *Preimage of a point of* $\mathrm{Mod}_{g,n}$ *under the forgetful map consists of finitely many equivalence classes of metrics.*

There are certain exceptional cases when the forgetful map is algebraic. In these cases its degree an be computed, and our conjecture is known to be true in these cases. Besides this, it is proved only in the special case when S is the sphere and $n = 4$, [2].

References

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