**Meromorphic functions with *a*-points in disjoint sets**


Alexandre Eremenko
(joint work with W. Bergweiler and A. Hinkkanen)
Purdue University, West Lafayette IN



Complex Analysis and related topics, Lviv, June 28-July 1, 2021.
Dedicated to the 90th anniversary of A. A. Goldberg


M.C. Escher

Value distribution theory studies the distribution of $a$-points, which are solutions of $f(z) = a$, for a meromorphic function $f : \mathbb{C} \to \overline{\mathbb{C}}$ and for various points $a$ in the Riemann sphere.
For two values of $a$, the $a$-points can be arbitrarily assigned, and Nevanlinna noticed in 1966 that one cannot arbitrarily assign $a$-points for three values of $a$. So one can ask for a condition on three sequences to be 0, 1 and $\infty$-points of some meromorphic function.
Known results by Rubel and C.-C. Yang, Ozawa, Winkler and others indicate that it is probably hopeless to look for a necessary and sufficient condition. For example, if $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are arbitrary sequences tending to infinity, then at least one of the three pairs

$$((a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty}), \; ((a_n)_{n=2}^{\infty}, (b_n)_{n=1}^{\infty}), \; ((a_n)_{n=3}^{\infty}, (b_n)_{n=1}^{\infty})$$

is not a $0 - 1$ sequence of any entire function.

So it seems reasonable to modify the problem:

*Let $A, B, C$ be three disjoint sets in the complex plane. Does there exist a meromorphic function whose all zeros are in $A$, 1-points are in $B$ and poles are in $C$?*

Besides intrinsic interest, this problem has a relation to control theory.
We restrict ourselves to the simplest setting when the sets $A, B, C$ are finite unions of rays from the origin or sectors.
The subject of meromorphic functions with $a$-points on rays seems to have some mysterious connection with analytic theory of linear differential equations, as we will see in examples below.

Two principal classical results are the following:

**Theorem of Edrei**. *If all zeros and* $1$*-points of an entire function* $f$ *belong to finitely many rays then the order of* $f$ *is finite. It is at most* $\pi/\omega$*, where* $\omega$ *is the smallest angle between two adjacent rays.*

This is remarkable since a condition on arguments of $a$-points implies a growth restriction.

**Theorem of Biernacki and Milloux**. *For a transcendental entire function of finite order, it is impossible for zeros to have a limiting direction, and for* $1$*-points to have a different limiting direction.*

(We say that a sequence $z_k$ has a limiting direction $\theta$ if $\arg z_k \to \theta$.)

Combining these two theorems we conclude

**Corollary**. *There is no transcendental entire function whose zeros lie on a ray, while* 1*-points belong to another ray.*

We notice that in Edrei's theorem it is important that $a$-points belong to rays rather than just have limiting directions: for any two distinct directions $\theta_1$ and $\theta_2$, one can construct an entire function *of infinite order* whose zeros have limiting direction $\theta_1$ and 1-points have limiting direction $\theta_2$.
This also shows that the a priori restriction of finite order is essential in the Biernacki–Milloux theorem and in the Corollary.

Next we consider the case of 3 rays from the origin.

**Theorem 1.** *Suppose that zeros of a transcendental entire function belong to a ray $L_0$, and 1-points belong to the union of two rays $L_{-1}$ and $L_1$ both distinct from $L_0$, and suppose that the numbers of zeros and 1-points are both infinite. Then*

$$\angle(L_0, L_1) = \angle(L_0, L_{-1}) < \pi/2.$$

It is interesting that there are indeed examples of such functions, with any angle $\alpha = \angle(L_0, L_1) \leq \pi/3$ and $\alpha = 2\pi/5$. It is unknown whether other angles $\alpha \in (\pi/3, \pi/2)$ can occur.

These examples arise from a simple differential equation

$$-y'' + x^m y = \lambda y, \quad x > 0.$$

If $y_0(x,\lambda)$ is a solution which tends to 0 as $x \to +\infty$, and normalized by its asymptotics as $x \to +\infty$, and $f(\lambda) = y_0(0,\lambda)$, then one can show that

$$\frac{\omega^{-1/2} f(\omega^{-2}\lambda) - \omega^{1/2} f(\omega^2 \lambda)}{f(\lambda)}$$

is an entire function with the stated poroperty. Here

$$\omega = e^{\pi i \alpha}, \quad \alpha = \frac{2\pi}{m+2}.$$

With few trivial exceptions, zeros and 1-points of entire functions cannot belong to two distinct lines.

**Theorem 2.** *Suppose that all zeros of an entire function lie on a line $L_0$, and all 1-points lie on a different line $L_1$ crossing $L_0$. Then $f(z) = e^{az+b}$ or $f(z) = 1 - e^{az+b}$, or a polynomial of degree at most 2.*

**Theorem (I. N. Baker, T. Kobayashi)** *Suppose that all zeros of an entire function lie on a line $L_0$ and all 1-points lie on a different line $L_1$ which is parallel to $L_0$. Then $f(z) = P(e^{az})$ for some polynomial $P$.*

Concerning meromorphic functions, we first state a generalization of Edrei's theorem whose assumption involves only arguments of 1-points, and the conclusion is that the order is finite:

**Theorem 3.** *Suppose that zeros, poles and* $1$*-points of a meromorphic function belong to finitely many rays, and on each of these rays one of the values* $0, 1, \infty$ *is omitted. Then the order is finite and does not exceed* $\pi/\omega$*, where* $\omega$ *is the smallest angle between the adjacent rays.*

All previous generalizations of Edrei's theorem to meromorphic functions involved an extra condition of the type that the function or its derivative has some deficient value.

Now we return to the original question, when zeros, poles and 1-points can belong to disjoint rays.

**Theorem 4.** *Let $L_0, L_1, L_\infty$ be three distinct rays from the origin. Let $f$ be a transcendental meromorphic function with all but finitely many $a$-points on $L_a$, where $a \in \{0, 1, \infty\}$. Then the rays must be equally spaced, that is the angle between any two of them equals $2\pi/3$.*

Such functions indeed exist, and they are almost completely described in the next theorem:

**Theorem 5.** *Let* $L_0, L_1, L_\infty$ *be equally spaced rays, and* $f$ *is a meromorphic function whose a-points lie on* $L_a$*, for* $a \in \{0, 1, \infty\}$*. Then* $f$ *is a solution of the Schwarz differential equation*

$$\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 = e^{3\theta i} z R(z^3),$$

*where* $\theta$ *is the argument of one of the rays, and* $R$ *is a real rational function with* $0 < R(\infty) < +\infty$*.*

There is no simple description of rational functions $R$ for which this differential equation has a meromorphic solution, but it is known that such rational functions exist with any given number of poles. We notice that our examples of entire functions with zeros on a ray and 1-points on two rays distinct from the first also come from the analytic theory of differential equations.

The simplest example illustrating Theorem 5 is the differential equation

$$\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 = 2z,$$

whose solutions are ratios of Airy functions: $f = w_1/w_2$, where $w_1, w_2$ are linearly independent solutions of

$$w'' + zw = 0.$$

These ratios can indeed be chosen so that zeros 1-points and poles lie on equally spaced raays

$$\{z : \arg z = 2\pi k/3\}, \quad k \in \{0, 1, 2\}.$$

All previous results have no a priori assumption of finiteness of order. If one makes such an assumption, then another type of generalization of Biernacki's theorem is possible, with rays replaced by sectors.

**Theorem 6.** *Let $S_0$ and $S_1$ be closed sectors of opening angles at most $\pi$, satisfying $S_0 \cap S_1 = \{0\}$. If $f$ is a transcendental entire function of finite order whose almost all zeros belong to $S_0$ and almost all 1-points belong to $S_1$, then*

$$f(z) = \int_0^z p(\zeta)e^{q(\zeta)}d\zeta + c,$$

*where $p$ and $q$ are polynomials, and $c$ is a constant.*

Notice that such functions $f$ also satisfy Schwarz differential equations.

The conclusion does not hold if one of the sectors has opening greater than $\pi$. For example, let

$$f(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right), \quad a_n > 0, \quad a_n \sim n^{1/\rho},$$

where $\rho \in (0, 1)$. This is an entire function with positive zeros, and arguments of 1-points accumulate to the directions $\arg z = \pm\pi(1 - 1/2\rho)$.

Here are some corollaries:

**Corollary 1.** *Let $S_0, S_2$ be closed sectors of opening angles $\theta_0, \theta_1$ and $S_0 \cap S_1 = \{0\}$. If*

$$\min\{\theta_0, \theta_1\} < \pi/2 \quad \textit{and} \quad \max\{\theta_0, \theta_1\} < \pi,$$

*then there is no transcendental entire function of finite order with almost all zeros in $S_0$ and almost all 1-points in $S_1$.*

**Example.**

$$f(z) = \frac{2}{\sqrt{\pi}} \int_0^z \zeta^2 \exp(-\zeta^2) d\zeta + 1/2$$

has almost all zeros in $\{z : |\arg z| \leq \pi/4\}$ and almost all 1-points in $\{z : |\arg a - \pi| \leq \pi/4\}$.

**Corollary 2.** *Let $S$ be a closed sector of opening angle less than $\pi/3$ and $H$ is a closed half-plane such that $H \cap S = \{0\}$. If $f$ is a transcendental entire function of finite order with almost all zeros in $S$ and almost all 1-points in $H$, then $f(z) = P(z)e^{az}$, where $P$ is a polynomial and $a \in \mathbb{C}$.*

**Example.**

$$f(z) = \int_0^z (a\zeta^3 + b\zeta)\exp(-\zeta^3)d\zeta + 1/3,$$

for an appropriate choice of $a$ and $b$ has almost all zeros in $\{z : |\arg z| \leq \pi/6\}$ and almost all 1-points in the left half-plane.

To illustrate our principal method, I sketch the proof of a special case of Corollary 1:

*There is no entire function of finite order, except polynomial of degree* 1*, whose zeros lie on a ray* $L$ *and* 1*-points lie in a closed sector* $S$ *of opening* $< \pi$ *such that* $L \cap S = \{0\}$.

Main idea of the proof. For simlicity we consider only the case when the order $\rho > 0$.
Then there exists a sequence $0 < r_k \to +\infty$ of *Pólya peaks* with the property that

$$\log M(tr_k) \leq (1+\epsilon)t^{\rho} \log M(r_k), \quad \epsilon < r < 1/\epsilon.$$

Using such a sequence, we define subharmonic functions

$$u_k(z) = \frac{\log |f(r_k z)|}{\log M(r_k)}, \quad v_k(z) = \frac{\log |f(r_k z) - 1|}{\log M(r_k)}.$$

These functions are uniformly bounded on compacts, and one can show that after selecting a subsequence of $r_k$ they tend to some subharmonic limits $u$ and $v$.
It is easy to see that each of these limits is not identical zero, and they have the following properties:

a) $\max\{u(z), v(z)\} = u^+(z) = v^+(z) \leq |z|^\rho,\ z \in \mathrm{C}$,

b) $u(0) = v(0) = 0$,

c) $u$ is harmonic in $\mathrm{C} \backslash L$ and $v$ is harmonic in $\mathrm{C} \backslash S$.

We will show that such functions cannot exist.

Case 1. $u$ is harmonic in C. Then we conclude from a) that $\rho$ is a positive integer, and

$$u(z) = cr^\rho \cos(\rho(\theta - \theta_0)).$$

Then a) implies that $v = u^+$, but this $v$ cannot be harmonic outside a sector of opening $< \pi$.

Case 2. $u$ is not harmonic in C.
Then we claim that $u(z) < 0$ in $D := \mathrm{C}\backslash S$.

Proof of the claim. Suppose first that, if $u \geq 0$ in $D$, then $v \leq u$ in $D$. If $v(z_0) < u(z_0)$, for some $z_0 \in D\backslash L$, then this inequality persists on an open set since both functions are continuous in $D\backslash L$. Then it follows from a) that $u = 0$ on this open set. But $u$ is harmonic on a dense subset of C, so we sould have $u = 0$ which is a contradiction with a).
If $u = v$ in $D$, then $u$ is harmonic in C, the case we considered before.

Suppose now that $u(z_0) < 0$ for some $z_0 \in D$ but $u(z_1) > 0$.
By upper semi-continuity we have $u(z) < 0$ in some neighborhood of $z_0$, thus $v(z) = 0$ in this neighborhood.
But $v$ is harmonic in $D$, so we conclude that $v(z) = 0$ in $D$ by uniqueness.
So then we have a contradition at the point $z_1$, where we must have $u(z_1) = v(z_0)$ by property a).
This shows that $u$ must be negative in $D$ which proves the claim.

The proof is concluded by the following elementary and useful

**Lemma.** *Let $u$ be a subharmonic function in a neighborhood of* 0*, and $u(0) = 0$. Then $u$ cannot be strictly negative in a sector of opening $> \pi$.*
*Proof.* Suppose wlog that the sector is

$$S = \{z : |\arg z| < \alpha\}, \quad \alpha > \pi/2.$$

Then $u$ has a hamonic majorant $-cr^{\rho}\cos(\rho\theta)$, where $\rho = \pi/(2\alpha)$ while in the complement of $S$ it has a harmonic majorant $c_1 r_1^{\rho}\cos(\rho_1(\theta - \pi)$. Since

$$\rho_1 = \pi/(2(\pi - \alpha)) > \rho,$$

we obtain that integrals of $u$ over small cirvles are negative, which contradicts the average property.