Entire functions with radially distributed zeros and 1-points, and some functional equations

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Let f be an entire function with all zeros positive. Can it happen that all solutions of $f(z) = 1$ lie on two rays from the origin, distinct from the positive ray? It is easy to show that these two rays must be symmetric with respect to real line and make equal angles $\alpha < \pi/2$ with the positive ray.

Examples of such entire functions really exist for $\alpha < \pi/3$ and for $\alpha = 2\pi/5$ [1], [2].

Do there exist such entire functions with $\alpha \in (\pi/3, \pi/2) \backslash \{2\pi/5\}$?

These examples in [1], [2] are "special functions" in the sense that they are related to simple linear differential equations, and satisfy some strange functional equation.

The simplest example, for $\alpha = 2\pi/5$ is a solution of the following functional equation:

$$f(\omega z)f(\omega^{-1}z) = f(z) - 1, \quad \omega = \exp(2\pi i/5).$$

Very little is known about entire solutions of this functional equation [3].

Is an entire solution unique, up to rotation of z?

References

[1] W. Bergweiler, A. Eremenko and A. Hinkkanen, Entire functions with two radially distributed values, Math. Proc. Cambridge Phil. Soc., to appear.

[2] A. Eremenko, Entire functions, PT-symmetry and Voros's quantization scheme, arXiv:1510.02504

[3] Y. Sibuya, Non-trivial entire solutions of the functional equation $f(\lambda) + f(\omega\lambda)f(\omega^{-1}\lambda) = 1,\ \omega^5 = 1$, Analysis 8 (1998), 271–295.