Real fish, real numbers, real jobs

Article *in* The Mathematical Intelligencer · June 1999

DOI: 10.1007/BF03024838

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Real Fish, Real Numbers, Real Jobs

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There's good times ahead for biologists. The electronics revolution will plateau in the early decades of the new millennium, and quantum computing, gene manipulation, and pet cloning will kick in. We'll see bumper crops of biology graduates pouring into expanded university departments, research institutes, and biotech companies. Some decades later, however, with hemoglobin-monitoring wrist watches and desktop chicken mutators as plentiful a nuisance as mobile phones and laptops are now, the spotlight of public attention and funds will inevitably shift to consumer applications. Where will that leave the ranks of theoretical research biologists amassed by universities?

My advice is to think now about saving the job prospects of future generations of academics by reconsidering and expanding the entire subject. That's where, I humbly suggest, *toaster fish* come in, and where mathematicians can provide some friendly help.

A toaster fish is of course a creature with the head of a toaster (the pop-up kind) and the body of a fish. Thus Figure 1 shows a typical toaster fish.

Toaster fish come in (at least) two different varieties—the North American kind, which operate on 120 volts, and the European kind, which use 220 volts. Already I hear a lot of you asking the obvious question: Can North American and European toaster fish mate? This is typical of the important new research problems in the brave new world of biology I'm proposing.

Biology needs to embrace the study of life in all its possibilities. There is no need to restrict attention to those creatures actually living, or having lived, or with the potential of living on planet earth and its immediate environs. These creatures are but a happy cross section of all possible life forms! By widening our consideration to *arbitrary* creatures, we can formulate new theories, create intellectual challenges, unleash our wildest imagination; and best of all, create jobs.

Let's call this emerging field *Life Theory.* Toaster fish, baby universes, and slimy galactic superoctupi will all be embraced by this far-sighted new intellectual endeavour.

But it won't be easy. There'll be colleagues who ridicule Life Theory. Those who call it a "theory about nothing" or "quasi-religious speculation on things which don't exist." Already one of my friends protests that Figure 1 amounts to little more than a poetic idea, since I have not "specified the internal workings" of the toaster fish figured.

Fortunately, mathematicians have already encountered and triumphed over similar obstacles in their historic struggle to place real numbers squarely in the centre of all modern mathemati-

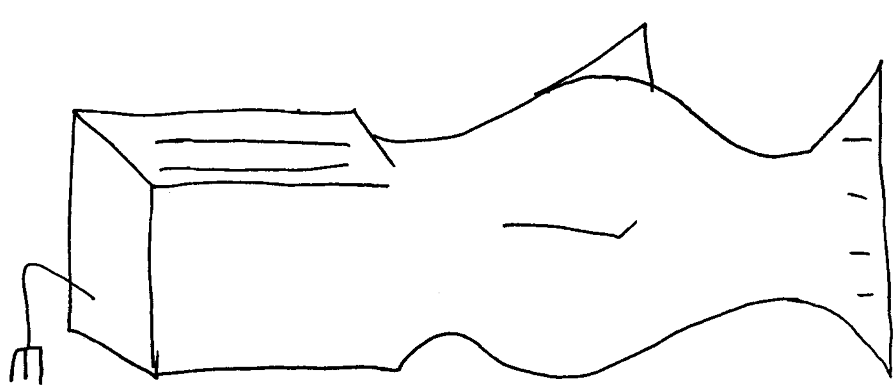


Figure 1. A toaster fish.

cal discourse. The Modern Biologist can gain from a careful study of this important achievement, which we turn to now with an eye to the main principles.

A Short History of Numbers and the Importance of Terminology

The Neanderthals gave us 1, 2, and 3. The ancients added on 4, 5, 6, and so on, discovered zero and negative numbers, and employed rationals like $\frac{22}{3}$. They discovered that other useful quantities like $\sqrt{2}$ and π were not of this form and could only be specified by algorithms, which subsequently allowed writing them in binary expansions like

$$\sqrt{2} = 1.011010100...$$
$$\pi = 11.00100100...$$

The Europeans added some more useful numbers: complex numbers like $5 + 2i$, and infinitesimal numbers like $1 + 3dx$. It wasn't clear exactly what these numbers were, but their use in the calculus was undisputed, so they remained. Until the nineteenth century, that is, when new standards of rigour revealed that "an infinitely small quantity" was only a grammatical construction bereft of any real mathematical meaning. So infinitesimals were abandoned. But complex numbers were justified and constructible irrationals abounded, so we still had numbers galore for every conceivable problem.

Then came the greatest, the boldest leap of all. We introduced numbers so abstract that they couldn't possibly be needed for any concrete problem. How? We considered binary expansions that couldn't be considered—by definition! Although this may sound contradictory, it is not. Just consider a slimy galactic superoctupus with an infinitely large brain, an infinite number of arms, and an inclination to wave them all simultaneously and independently either up or down. Looking at the configuration of arms at some random time gives a generally unconstructible sequence. As an aside, I mention that such a creature would be a useful friend, being able to solve the Goldbach Conjecture or the Riemann Hypothesis simply by computation. And in an infinitesimally small amount of time to boot.

More mundanely, an uncomputable binary expansion is simply a binary expansion that is not computable. It does seem curious that such a simple idea had eluded all the great thinkers until the latter half of the nineteenth century. To envision such a number we proceed as follows. Randomly write down a string of zeroes and ones; a dozen usually suffice, but if the audience is skeptical I like to include more; and then add three dots. Thus

$$\alpha = 1.011010111101...$$

shows a typical unconstructible number. These new objects, when added on to the collection of rationals and constructible irrationals, gave us a whole new domain of mathematical discourse.

Now a key point. What did we call this new number system consisting of largely chimerical creations of our imagination? Did we call them Arbitrarials? or Way-out numbers? No. In a flash of brilliance any PR person would be proud of, we called them *real numbers*, and designated them by the solid-looking $\mathbb{R}$.

This inspired terminology is worth emulating. I suggest you call the new area of Modern Biology concerned with toaster fish, refrigerator fish, and so on the field of *real fish*. Those real fish actually found in the solar system will then have a special designation, subtly reinforcing the idea that it is unnatural to consider them separately; let's say *observable real fish*. How much more of a bother it is to consider the latter.

"Let α be an observable real fish," you begin your lecture. Where's it from? Is it green? What does it eat? someone immediately asks. The preferable "Let α be a real fish" protects you from such unnecessary questions, which you can't answer anyway!

In mathematics, restriction to computable real numbers is a decided nuisance. First one must worry about what "computable" means. Which machine? Which language? About whether or not programs halt. Simple operations like adding or multiplying two numbers give us headaches. Real numbers free us from these hassles, which we prefer to leave to the computer scientists anyway.

Training the Young

How do we get future generations to take the validity of real numbers for granted? We indoctrinate them early in their careers when they are eager but impressionable undergraduates. Here's how we do it. First we soften them up with a "Constructing the Real Numbers" blurb in their first calculus course. Needless to say we don't really construct real numbers as they are by definition unconstructible. But the phrase sticks in their minds long after the details are forgotten.

Sometime later we expose them to the definition of a 1:1 correspondence, Cantor's wonderful proof that there is no list of all binary sequences, and voilà! With a bit of hand-waving about what a set actually is, a vast and glittering new universe of ordinals, cardinals, hierarchies of infinities, and continuum hypotheses opens up before their eyes. The confusion and apprehension of millenia cast aside, our students are soon fearlessly manipulating arbitrary unions of arbitrarily many powers of uncountable ordinals.

It's a heady experience, this first grasping hold of infinity.

But be warned; there will be some students who won't buy this. After all, young people are a tad critical at this age. We explain to them that if they want to get into "foundational questions" then they had first better master the "predicate calculus of first-order logic." Having barely mastered the ordinary calculus, they seem to find this a convincing argument.

A further point. We don't start with younger students since they get too easily confused. They want to know specifics like "How does one add or multiply real numbers?" In fact one *can't* add or multiply real numbers, but of course this shouldn't and doesn't prevent us from using these operations on a daily basis. Immature learners are likely to get stuck by these subtleties.

The snag is the following. Suppose we want to add the unconstructible real numbers 1.010101101000110 . . . and 1.100010010111001. . . . Clearly the first digits of the sum are 10.11, but the next digit requires more information. How much more information? Possibly just one more digit, possibly a trillion more digits, and here's the rub—possibly infinitely many more digits! The complementary pattern of zeroes and ones might continue to infinity, but if so we'll never know it. This prevents us from ever specifying a procedure to add real numbers. One might hope that trying some different definition of the reals using continued fractions or Dedekind cuts or Cauchy sequences of rationals might solve the problem, but it doesn't. It just transforms it to an equivalent one.

But here's where religion, or Modern Biology, comes to our aid. It may be impossible for us to know whether a given complementary pattern of zeroes and ones continues indefinitely, but it is clearly not impossible for God, or a slimy intergalactic superoctopus, to know. So we are blessedly free to go on talking about adding arbitrary real numbers.

Convincing the Professoriate

This is a question primarily of critical mass. Once enough of the Academy accept and profit by the new bounds of the discipline, a critical momentum will prevent opponents from stemming the tide. First of all a leading luminary must be taken on board. In our case, that was David Hilbert, whose oft quoted "No-one shall expel us from the paradise that Cantor has created for us" becomes our motto.[1]

Furthermore, we must convince our colleagues that opportunities for research are rife. In mathematics, the kind of free thinking that allows the pleasant contemplation of real numbers (I like to think of them as basking in the sun) also revels in more complicated abstractions. Unlike earlier more squeamish generations, we are quite comfortable with arbitrary functions from $\mathbb{R}$ to $\mathbb{R}$ (not just those given by formulae or computer programs but all the others too), with arbitrary operators on spaces of such functions, with arbitrary functionals on algebras of such operators and so on. Non-measurable functions, Banach-Tarski paradoxes, various versions of the Axiom of Choice, inaccessible cardinals—none of these ruffles our calm in the brave new world of twentieth-century mathematics.

More than just jobs are opened up here though. Our easy attitude to sets can be extended to other mathematical objects too. This liberal approach is captured by what we might call the *Axiom of Freedom*—an object is constructed, or specified (same thing), by simply stating it. To give but one of a multitude of possible uses of this principle, suppose you've finally constructed a quasi-barrelled inverted amorphous fibre scheme S and all that remains is to exhibit the space P of all coherent quadratic functors from S to its dual. In years gone by this would have required fretting about precise definitions and concrete realizations. These days, you need only say, "Let P be the space of all coherent quadratic functors from S to its dual." Voilà, the construction is complete.

Setbacks and Recovery

At the turn of the century, with mathematicians divided in controversy over the validity and meaning of Cantor's new theory and the role of real numbers, disaster struck. In one of the most embarrassing intellectual deflations of all time, the entire new theory came crashing down with the discovery of fundamental new paradoxes. These weren't just the niggling kind of inconsistencies acknowledged from the start even by Cantor, they were the full-blown $\pi = \sqrt{2}$ type of contradictions that ruin one's entire week. A dark time indeed, but ultimately only a minor setback. After some hesitant and unconvincing attempts to shore up the subject, we now see that worrying about such issues is largely a waste of time. Foundations are not really all that necessary.

If pressed by philosophical types, we have a fall-back position. Mathematics is based on Axioms, and the Axioms for Set Theory were spelt out by Zermelo and Fraenkel. This is the Formalist position, also pushed by Hilbert, which got rather badly bruised by Gödel's results in the 1930s. As a result, most of us closet Platonists aren't so keen on this official explanation, but who really wants to get embroiled in philosophical hair-splittings? An intellectual discipline reflects the spirit and values of the times, and these are times of pragmatism and expediency.

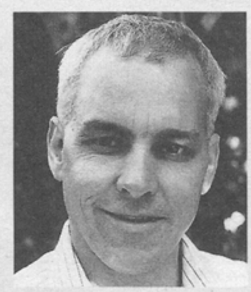
If pressed by our own kind, by mathematicians who try to point out the imprecision of our cherished fundamental concepts, we respond politely. We allow them their say, listen patiently, and then point out that they're basically only repeating Brouwer's critique of 1918.

And what about those who obstinately refuse to understand? Who keep on insisting that you explain "What *exactly* is a real fish?" Sigh. These people are quibblers—philosophical quibblers at that—and they are best ignored. After all, you've got a job to do.

Looking to the Future

As pragmatism and expediency don't look to be departing the modern world anytime soon, I'm confident that the lessons mathematicians have learnt can be fruitfully applied to biology. Don't expect to convince your contemporaries about the importance of real fish. Recruit a few top names,

AUTHOR



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[1]Naturally we don't advertise, for example, that Gauss wrote, "I protest—against the use of infinite magnitude as if it were something finished; this use is not admissible in mathematics. The infinite is only a *façon de parler*; one has in mind limits approached by certain ratios as closely as desired while other ratios may increase indefinitely."

work on the future generations, keep the employment prospects firmly in view, and Life Theory will take its rightful place in the scientific world.

As for me, this little advice-giving has got me to thinking. Mathematicians are themselves a bit hard pressed these days for employment. The huge amount of time and effort required to rewrite modern mathematics using only computable numbers, computable functions, computable everythings would keep us busy for decades. What an unpleasant lot of hard work that would be.

But hard work means jobs, grants, and pensions! Maybe Aristotle, Archimedes, Newton, Euler, Gauss, Kronecker, Borel, Lebesgue, Poincaré, Weyl, Brouwer, and the others were right. There *is* something fishy about real numbers.

Going with the Flow

Excerpt from a BBC interview with Sir James Lighthill, the great applied mathematician. The interview was conducted by the biologist Lewis Wolpert and produced by Alison Richards.[1]

Wolpert: Now most of your work has been in fluids. Is there something about fluids that appeals to you?

Lighthill: Aha, yes, I think so! I have a sort of general pleasurable feel about fluids and, of course, I'm very interested in flight, and although I worked entirely on aeronautical flight in those days, I subsequently did very comprehensive studies of animal flight—birds, bats and insects—during my later period in Cambridge, working with the zoology department there. And my hobby is swimming; I have a great deal of interest in the ocean—ocean waves, ocean currents, ocean tides—and so I enjoy observing all that when I swim. And then I have a fellow feeling for the swimming animals, and I've written papers about almost all varieties of swimming fishes and invertebrates, and quite a lot of work on micro-organism locomotion.

W.: Part of your passion for fluids is swimming.

L.: Yes, indeed.

W.: Do you swim a lot?

L.: Yes, I do a three-mile swim every weekend just to keep fit.

W.: And in the holidays?

L.: In the holidays I always do each year an adventure swim, which I do, partly because it's good for all of us to have an adventure every so often, but partly because when I was at Farmborough I was working with test pilots, and I was conscious that they were actually depending on the scientific work that was done; they staked their lives on the correctness of the science. I've done a lot of work on ocean waves and tides and currents, and I feel I understand them well enough to be quite prepared to swim in them, because with my theoretical knowledge, supplemented by an immense amount of experience in swimming in these conditions, I can swim safely, and have an exciting adventure in the process. So I do this, usually choosing swims where there are quite difficult currents to deal with. Sometimes swims around islands, sometimes swims between one island and another.

W.: Like what?

L.: Well one of my famous swims is the one around Sark which I've done five times, and one of them was during a south-westerly gale which was the one that actually caused the Fastnet disaster. So one needed quite a lot of nerve and stamina to complete that swim on that day, but it really was rather an exciting experience. But I've swum between two of the Azores which have quite a strong current between them. I've swum round an actively erupting volcano, namely Stromboli, and watched eleven eruptions from the side where you can see the volcano, where incidentally, the water is the temperature of a hot bath because that's the side the lava comes into the sea. And I've swum round Lundy, and my most recent swim was round Ramsay Island where there are exceptionally strong currents off the south-west coast of Pembrokeshire.

W.: Do you actually use your knowledge of waves and tides in order to do it?

L.: Oh enormously, yes. I mean during this Fastnet swim I was constantly having to sort of add up vectorially my swimming velocity and the current velocity, and the wave drift due to these very powerful waves. It was rather interesting. I was really having to swim at right angles to the direction I wanted to go, which you often have to do, of course.

W.: [Laughs] I don't think many of us would recognize that.

On 17 July 1998, James Lighthill was about to complete his sixth swim around Sark when he died—not of drowning, it turns out, but of mitral valve disease.

[1]The text appears in their book *Passionate Minds*, Oxford University Press, 1997. The passage quoted is on pp. 62–63. Used by permission of Oxford University Press.