

Zeros of differential polynomials in real meromorphic functions

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Abstract

We investigate when differential polynomials in real transcendental meromorphic functions have non-real zeros. For example, we show that if g is a real transcendental meromorphic function, $c \in \mathbf{R} \setminus \{0\}$ and $n \geq 3$ is an integer, then $g'g^n - c$ has infinitely many non-real zeros. If g has only finitely many poles, then this holds for $n \geq 2$. Related results for rational functions g are also considered.

1 Introduction and results

Our starting point is the following result due to Sheil-Small [15] which solved a longstanding conjecture.

Theorem A *Let f be a real polynomial of degree d . Then $f' + f^2$ has at least $d - 1$ distinct non-real zeros which are not zeros of f .*

In the special case that f has only real roots this theorem is due to Prüfer [14, Ch. V, 182]; see [15] for further discussion of the result.

The following Theorem B is an analogue of Theorem A for transcendental meromorphic functions. Here “meromorphic” will mean “meromorphic in the complex plane” unless explicitly stated otherwise. A meromorphic function is called real if it maps the real axis \mathbf{R} to $\mathbf{R} \cup \{\infty\}$.

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Theorem B *Let f be a real transcendental meromorphic function with finitely many poles. Then $f' + f^2$ has infinitely many non-real zeros which are not zeros of f .*

Theorem B is a special case of [3, Theorem 1.3]. The result that $f' + f^2$ has infinitely many non-real zeros if f is a real entire transcendental function follows from the main theorem of [4].

A corollary of Theorem A is the following result.

Corollary 1 *Let f be a real polynomial of degree d and $m \geq 2$ an integer. Then $f' + f^m$ has at least $d(m-1) - 1$ distinct non-real zeros which are not zeros of f . In particular, $f' + f^m$ has at most $d + 1$ real zeros.*

In fact, putting $g(z) = f(w)^{m-1}$ where $w = z/(m-1)$ we have

$$g'(z) + g(z)^2 = f(w)^{m-2} (f'(w) + f(w)^m)$$

so that Corollary 1 follows from Theorem A applied to g , since g has degree $d(m-1)$.

The same argument yields the following corollary to Theorem B.

Corollary 2 *Let f be a real transcendental meromorphic function with finitely many poles and $m \geq 2$ an integer. Then $f' + f^m$ has infinitely many non-real zeros which are not zeros of f .*

We note that Theorem B does not hold for meromorphic functions with infinitely many poles, a simple example being $f(z) = -\tan z$. Similarly Theorem A fails for rational functions.

In this paper we consider to what extent Corollary 1 holds for rational functions, and Corollary 2 for meromorphic functions f with infinitely many poles. We recall that Hayman [6, Corollary to Theorem 9] showed that if f is a transcendental meromorphic function and $m \geq 5$, then $f' + f^m$ has infinitely many zeros. In fact, his proof shows that $f' + f^m$ has infinitely many zeros which are not zeros of f . (This can also be seen from the main result of [2] and formula (1) below.)

Mues [11] showed that Hayman's result remains valid for $m = 4$, and the first and second author [2] proved that this also holds for $m = 3$. We also note that if f is rational and $f' + f^m$ has no zeros for some $m \geq 3$, then f is constant.

Theorem 3 *Let f be a real rational function of degree d and $m \geq 5$ an integer. Then $f' + f^m$ has at least $(m - 4)d$ distinct non-real zeros which are not zeros of f . In particular, $f' + f^m$ has at most $4d$ real zeros.*

If m is odd, then $f' + f^m$ has at least $(m - 3)d$ distinct non-real zeros which are not zeros of f so that $f' + f^m$ has at most $3d$ real zeros.

Theorem 4 *Let f be a real transcendental meromorphic function and $m \geq 5$ an integer. Then $f' + f^m$ has infinitely many non-real zeros which are not zeros of f .*

Theorem 4 improves Hayman's result in the case that f is real. We will show by examples that the restriction $m \geq 5$ in this theorem is best possible.

If $g(z) = 1/f(w)$ where $w = cz$, then

$$f'(w) + f(w)^m = c^{-1}g(z)^{-m}(c - g(z)^{m-2}g'(z)), \quad (1)$$

and we obtain the following results from Theorems 3 and 4.

Corollary 5 *Let g be a real rational function of degree d and $n \geq 3$ an integer. Then for every real $c \neq 0$ the equation $g^n g' = c$ has at least $d(n - 2)$ distinct non-real solutions.*

Corollary 6 *Let g be a real transcendental meromorphic function and $n \geq 3$ an integer. Then for every real $c \neq 0$ the equation $g^n g' = c$ has infinitely many non-real solutions.*

The condition $n \geq 3$ is best possible in order to conclude that there are non-real solutions. However for polynomials and transcendental meromorphic functions with finitely many poles we have

Theorem 7 *Let g be a real polynomial of degree d and $n \geq 2$ an integer. Then for every real $c \neq 0$ the number of distinct non-real solutions of the equation $g^n g' = c$ is at least $d(n - 1) - 1$ if n is even and at least $d(n - 1)$ if n is odd. In particular, this equation has at most $2d$ real solutions.*

Theorem 8 *Let g be a real transcendental meromorphic function with finitely many poles and $n \geq 2$ an integer. Then for every real $c \neq 0$ the equation $g^n g' = c$ has infinitely many non-real solutions.*

Examples show that if $n = 1$, then the equation $g^n g' = c$ need not have non-real solutions.

For our last result, we return to the value distribution of $f' + f^m$. Hayman [6] proved not only that if f is a transcendental meromorphic function and $m \geq 5$ then $f' + f^m$ has infinitely many zeros, but that under these conditions $f' + f^m + c$ has infinitely many zeros for any $c \in \mathbf{C}$. Examples show that when m is 5 or 6 and $c \in \mathbf{R} \setminus \{0\}$ then $f' + f^m + c$ may fail to have non-real zeros, with f real, transcendental and meromorphic.

Theorem 9 *Let f be a real meromorphic function and set $G = f' + f^m + c$, where $c \in \mathbf{R} \setminus \{0\}$ and $m \in \mathbf{N}$.*

Suppose that f is transcendental. If $m \geq 7$ then G has infinitely many non-real zeros which are not zeros of f . The same conclusion holds for $m \geq 4$ if $\overline{N}(r, f) = o(T(r, f))$ and thus in particular if f has finitely many poles.

Suppose finally that f is a non-constant rational function with $f' \neq -c$. If $m \geq 6$ then G has at least one non-real zero which is not a zero of f , and the same conclusion holds for $m \geq 3$ if f is a polynomial.

2 A result from complex dynamics

One of our main tools is a result from holomorphic dynamics. Recall that the Fatou set of a non-linear meromorphic function f is the set where the iterates $f^{\circ n}$ of f are defined and form a normal family. We say that $\zeta \in \mathbf{C}$ is a multiple fixed point of f (of multiplicity μ) if $f(z) - z$ has a multiple zero (of multiplicity μ) at ζ . This has to be slightly modified if $\zeta = \infty$. We say that ∞ is a multiple fixed point of f (of multiplicity μ) if $1/f(1/z)$ has a multiple fixed point (of multiplicity μ) at 0.

Lemma 10 *Let f be a non-linear rational or transcendental meromorphic function and let ζ be a multiple fixed point of f of multiplicity μ . Then there are $\nu := \mu - 1$ components U_1, U_2, \dots, U_ν of the Fatou set of f satisfying $f(U_j) \subset U_j$, $\zeta \in \partial U_j$ and $f^{\circ n}(z) \rightarrow \zeta$ for all $z \in U_j$ as $n \rightarrow \infty$. Each U_j contains at least one singularity of f^{-1} , the inverse function of f .*

In addition, the U_j can be labelled such that if $z \in U_j$, then

$$\arg(f^{\circ n}(z) - \zeta) \rightarrow \theta_j := \frac{-\arg f^{(\nu+1)}(\zeta) - \pi + 2\pi j}{\nu}$$

as $n \rightarrow \infty$ if $\zeta \in \mathbf{C}$, while if $\zeta = \infty$ then, for some $\theta_0 \in \mathbf{R}$,

$$\arg(f^{\circ n}(z)) \rightarrow \theta_j := \frac{\theta_0 + 2\pi j}{\nu}$$

as $n \rightarrow \infty$.

The domains U_j appearing in Lemma 10 are called *Leau domains*. Note that the singularities of f^{-1} are precisely the critical and asymptotic values of f . If f is rational, then the only singularities of f^{-1} are the critical values. In fact, the domains U_j then also contain critical points, but we do not need this result.

Lemma 10 is due to Fatou, and can now be found in every textbook on complex dynamics. Excellent introductions to complex dynamics are [10, 16]; see [10, §10] or [16, §3.5] for a proof and discussion of the results stated in Lemma 10. We note that [10, 16] and most other textbooks on complex dynamics treat only the case that f is rational, but the proofs extend to the case of transcendental meromorphic functions; see also [1, §4.3].

To demonstrate how dynamics works we begin with a simple direct proof of Corollary 1 found by the second author in 1989, after reading Sheil-Small's paper. This dynamical proof is also reproduced in [8].

Proof of Corollary 1. Put

$$F(z) = z - \frac{1}{f(z/(m-1))^{m-1}}. \quad (2)$$

Then $F'(z) = 0$ gives $f'(w) + f(w)^m = 0$ where $w = z/(m-1)$. We have

$$F(z) = z + cz^{-d(m-1)} + O(z^{-d(m-1)-1}), \quad z \rightarrow \infty,$$

where $c \in \mathbf{R} \setminus \{0\}$. This implies that ∞ is a multiple fixed point of f of multiplicity $\mu = d(m-1) + 2$. Let $U_1, U_2, \dots, U_{\mu-1}$ be the Leau domains at ∞ and let $\theta_1, \theta_2, \dots, \theta_{\mu-1}$ be as in Lemma 10. Each U_j contains a critical value and, as F is real, this critical value and the corresponding critical point can be real only if θ_j is a multiple of π . This is the case for at most 2 of the $\mu - 1$ values of j , and thus F has at least $\mu - 3 = d(m-1) - 1$ non-real critical points.

As the total number of zeros of $f' + f^m$ is dm we obtain the result. \square

The proof of Theorem 3 will use the same idea.

Proof of Theorem 3. Let $f(z) = P(z)/Q(z)$ where P has degree p and Q has degree q , so that $d = \max\{p, q\}$. We consider again the function F defined by (2). If $q < p = d$, then

$$F(z) = z + cz^{-(d-q)(m-1)} + O(z^{-(d-q)(m-1)-1}), \quad z \rightarrow \infty,$$

and the argument used in the proof of Corollary 1 shows that F has a multiple fixed point at ∞ and that the Leau domains associated to this fixed point contain at least $(d - q)(m - 1) - 1$ non-real zeros of F' .

Let ζ_1, \dots, ζ_k be the finite poles of f , with multiplicities s_1, \dots, s_k . Then $\zeta_j(m - 1)$ is a fixed point of multiplicity $s_j(m - 1)$ of F . Lemma 10 now yields that of the $s_j(m - 1) - 1$ Leau domains associated to ζ_j at most 2 can contain a real critical value, so at least $s_j(m - 1) - 3$ of these domains give rise to non-real zeros of F' .

Overall this leads to $\sum_{j=1}^k s_j(m - 1) - 3 = q(m - 1) - 3k$ non-real zeros of F' . If $q = d$ we thus have $d(m - 1) - 3k \geq d(m - 1) - 3d = d(m - 4)$ non-real zeros of F' . If $q < d$, then we obtain $(d - q)(m - 1) - 1 + q(m - 1) - 3k = d(m - 1) - 1 - 3k \geq d(m - 1) - 1 - 3(d - 1) = d(m - 4) + 2$ non-real zeros of F' .

If m is odd, then the number of Leau domains at the ζ_j and ∞ is odd. Lemma 10 shows that each of them, with at most 1 exception, contains a non-real critical value. The above argument then shows that F' has at least $d(m - 3)$ non-real zeros. \square

3 Proof of Theorem 4

Our second main tool is the following result of Pang [13] which already found many applications, see, for example, Zalcman's survey [18].

Lemma 11 *Let f be a meromorphic function with unbounded spherical derivative, and $\kappa \in (-1, 1)$. Then there exist sequences $z_j \in \mathbf{C}$ and $a_j > 0$ such that*

$$a_j^{-\kappa} f(z_j + a_j z) \rightarrow h(z), \quad j \rightarrow \infty, \quad (3)$$

uniformly on compact subsets of \mathbf{C} , where h is a non-constant meromorphic function with bounded spherical derivative. Furthermore, one can choose

$$z_j \rightarrow \infty \quad \text{and} \quad a_j \rightarrow 0, \quad j \rightarrow \infty. \quad (4)$$

The following result is an immediate consequence of the definition of the order of a meromorphic function, using the Ahlfors-Shimizu form of the Nevanlinna characteristic.

Lemma 12 *Let f be a meromorphic function with bounded spherical derivative. Then f is of order at most 2.*

Proof of Theorem 4. The proof is by contradiction, assuming that $f' + f^m$ has only finitely many non-real zeros which are not zeros of f .

The first step is to reduce the result to the case of functions with bounded spherical derivative. Suppose f has unbounded spherical derivative. We apply Lemma 11 with $\kappa = 1/(1 - m)$. Then $\kappa m = \kappa - 1$ and we obtain

$$a_j^{-\kappa m} f(z_j + a_j z)^m \rightarrow h(z)^m.$$

In $\mathbf{C} \setminus h^{-1}(\{\infty\})$ we have

$$a_j^{-\kappa m} f'(z_j + a_j z) = a_j^{-\kappa + 1} f'(z_j + a_j z) \rightarrow h'(z)$$

and

$$h'(z) + h(z)^m = \lim_{j \rightarrow \infty} a_j^{-\kappa m} (f'(z_j + a_j z) + f(z_j + a_j z)^m).$$

If $|\operatorname{Im} z_j/a_j| \rightarrow \infty$ then all zeros of $h' + h^m$ are zeros of h . This contradicts the result of Hayman [6, Corollary to Theorem 9] mentioned in the introduction; see also [2].

Thus $|\operatorname{Im} z_j/a_j| \not\rightarrow \infty$ and we may assume that $\operatorname{Im} z_j/a_j \rightarrow s \in \mathbf{R}$. Putting $x_j = \operatorname{Re} z_j = z_j - i \operatorname{Im} z_j$ we find that

$$a_j^{-\kappa} f(x_j + a_j z) = a_j^{-\kappa} f(z_j + a_j(z - i \operatorname{Im} z_j/a_j)) \rightarrow h(z - is).$$

We may thus assume that $z_j \in \mathbf{R}$ so that h is real, since otherwise we can replace z_j by x_j and $h(z)$ by $h(z - is)$. We obtain a non-constant real meromorphic function h with bounded spherical derivative. It follows from (4) that *all* non-real zeros of $h' + h^m$ are zeros of h . Theorem 3 implies that h is transcendental.

If f has bounded spherical derivative, then we put $h = f$. Again h is transcendental, but $h' + h^m$ may have finitely many non-real zeros which are not zeros of h .

As in the proof of Corollary 1 and Theorem 3 we consider the auxiliary function

$$F(z) = z - \frac{1}{h(z/(m-1))^{m-1}},$$

and note that poles of h give rise to multiple fixed points of F of multiplicity at least $m - 1$. Lemma 10 implies that at least 2 of the (at least 3) Leau domains associated to such a fixed point contain a non-real singularity of F^{-1} .

Theorem 1 from [2] implies that all non-real asymptotic values correspond to logarithmic singularities of the inverse function F^{-1} . But according to the Denjoy–Carleman–Ahlfors Theorem [12, §258], a function of order at most 2 can have at most 4 logarithmic singularities. Thus F has only finitely many non-real critical and asymptotic values, so we conclude that h has finitely many poles. We obtain a contradiction with Corollary 2. \square

4 Proof of Theorems 7 and 8

Proof of Theorem 7. Without loss of generality we can assume that $c = 1$. We consider the function

$$G(z) = z - \frac{1}{n+1}g(z)^{n+1}$$

and note that the critical points of G are exactly the solutions of $g(z)^n g'(z) = 1$.

If ζ is a zero of g of multiplicity s , then ζ is a multiple fixed point of G of multiplicity $s(n+1)$. Lemma 10 yields that the $s(n+1) - 1$ associated Leau domains contain at least $s(n+1) - 3$ non-real critical values. If n is odd and thus the number $s(n+1) - 1$ of Leau domains is odd, then we even obtain $s(n+1) - 2$ non-real critical values in these domains. Let ζ_1, \dots, ζ_k be the zeros of g , with multiplicities s_1, \dots, s_k . If n is odd then we find that G' has at least $(n+1) \sum_{j=1}^k s_j - 2k = d(n+1) - 2k$ non-real zeros. Since $k \leq d$ this implies that G' has at least $d(n-1)$ non-real zeros.

Now we consider the case that n is even. If ζ is a simple zero of g , then the previous argument based on Lemma 10 will only yield that $n-2$ of the n associated Leau domains contain a non-real critical value, which does not give any information in the most interesting case that $n=2$. We note, however, that if $g'(\zeta) < 0$, then $G^{(n+1)}(\zeta) = -n!g'(\zeta)^{n+1} > 0$. With U_j as in Lemma 10 we find that if $z \in U_j$, then

$$\arg(G^{\circ \ell}(z) - \zeta) \rightarrow \theta_j := \frac{-\arg G^{(n+1)}(\zeta) - \pi + 2\pi j}{n} = \frac{(2j-1)\pi}{n}$$

as $\ell \rightarrow \infty$. As n is even, $(2j-1)/n$ is never an integer and thus all n Leau domains at ζ contain a non-real critical value in this case.

Let K be the number of real simple zeros ζ of g for which $g'(\zeta) > 0$. Between two such zeros there must be a real simple zero ζ satisfying $g'(\zeta) < 0$ or a multiple zero of odd multiplicity. Let L be the number of real simple zeros ζ of g for which $g'(\zeta) < 0$ and let M be the number of real multiple zeros of odd multiplicity. Then

$$L + M + 1 \geq K. \quad (5)$$

We denote by N the total number of multiple zeros of g and by P the number of non-real simple zeros. Thus $K + L + P$ is the total number of simple zeros of g . We find that

$$d \geq K + L + P + 3M + 2(N - M) = K + L + P + M + 2N. \quad (6)$$

Denote by ξ_1, \dots, ξ_N the multiple zeros of g , with multiplicities t_1, \dots, t_N . The above considerations based on Lemma 10 show that there are at least $(n+1)t_j - 3$ non-real critical values of G contained in the Leau domains associated to ξ_j . Also, for a non-real simple zero of g each of the n associated Leau domains contains a non-real critical value of G . Overall the number ν of non-real critical points of G thus satisfies

$$\begin{aligned} \nu &\geq (n+1) \sum_{j=1}^N t_j - 3N + K(n-2) + Ln + Pn \\ &= (n+1)(d - K - L - P) - 3N + K(n-2) + Ln + Pn \\ &= (n-1)d + 2d - 3N - 3K - L - P. \end{aligned}$$

Using (6) and (5) we obtain

$$\begin{aligned} \nu &\geq (n-1)d + 2(K + L + P + M + 2N) - 3N - 3K - L - P \\ &= (n-1)d - K + L + 2M + N + P \\ &\geq (n-1)d - (L + M + 1) + L + 2M + N + P \\ &= (n-1)d - 1 + M + N + P. \end{aligned}$$

The conclusion follows since $M, N, P \geq 0$. \square

To prove Theorem 8 we cannot use the reduction to functions of finite order based on Lemma 7, because there exists a non-constant real entire function, namely $h(z) = z$, with the property that all solutions of the equation

$h'(z)h(z)^2 = 1$ are real. We need instead a direct proof that g is of finite order.

Proof of Theorem 8. In view of Corollary 6 it is enough to consider the case $n = 2$. We can assume without loss of generality that $c = 1$.

Suppose that the equation $g'(z)g(z)^2 = 1$ has finitely many non-real solutions. First we show that g has order at most 1. To accomplish this, we need the characteristic function in the upper half-plane as developed by Tsuji [17] and Levin and Ostrovskii [9] (see also [5]), and as used in [3]. For ψ meromorphic and non-constant in the closed upper half-plane $\text{Im } z \geq 0$ and for $t \geq 1$ let $\mathfrak{n}(t, \psi)$ be the number of poles of ψ , counting multiplicity, in $\{z : |z - it/2| \leq t/2, |z| \geq 1\}$, and set

$$\mathfrak{N}(r, \psi) = \int_1^r \frac{\mathfrak{n}(t, \psi)}{t^2} dt, \quad r \geq 1.$$

The Tsuji characteristic is

$$\mathfrak{T}(r, \psi) = \mathfrak{m}(r, \psi) + \mathfrak{N}(r, \psi),$$

where

$$\mathfrak{m}(r, \psi) = \frac{1}{2\pi} \int_{\sin^{-1}(1/r)}^{\pi - \sin^{-1}(1/r)} \frac{\log^+ |\psi(r \sin \theta e^{i\theta})|}{r \sin^2 \theta} d\theta.$$

We refer the reader to [3, 5, 9, 17] for the fundamental properties of the Tsuji characteristic, but note in particular that the lemma on the logarithmic derivative [9, p.332] (see also [5, Theorem 3.2, p.141]) gives

$$\mathfrak{m}(r, \psi'/\psi) = O(\log r + \log^+ \mathfrak{T}(r, \psi)) \quad (7)$$

as $r \rightarrow \infty$ outside a set of finite measure.

The following lemma is a direct analogue for g of a result of Hayman from [6].

Lemma 13 *We have*

$$\mathfrak{T}(r, g) = O(\log r), \quad r \rightarrow \infty.$$

Proof. We follow Hayman's proof as in [6], but using the Tsuji characteristic and in particular (7). Let

$$\phi(z) = \frac{1}{3}g(z)^3.$$

Then ϕ has finitely many poles, and $\phi' - 1$ has finitely many zeros in the open upper half-plane H , and so

$$\mathfrak{N}(r, \phi) + \mathfrak{N}\left(r, \frac{1}{\phi' - 1}\right) = O(1). \quad (8)$$

Milloux' inequality [7, Theorem 3.2, p.57] translates directly in terms of the Tsuji characteristic to give, using (7) and (8), outside a set of finite measure,

$$\mathfrak{T}(r, \phi) < \mathfrak{N}\left(r, \frac{1}{\phi}\right) - \mathfrak{N}_0\left(r, \frac{1}{\phi''}\right) + O(\log r + \log^+ \mathfrak{T}(r, \phi)) \quad (9)$$

in which $\mathfrak{N}_0\left(r, \frac{1}{\phi''}\right)$ counts only zeros of ϕ'' which are not multiple zeros of $\phi' - 1$. But all zeros of ϕ have multiplicity at least 3 and so are zeros of ϕ'' but not zeros of $\phi' - 1$, and consequently each such zero contributes 2 to $\mathfrak{n}\left(r, \frac{1}{\phi}\right) - \mathfrak{n}_0\left(r, \frac{1}{\phi''}\right)$ but at least 3 to $\mathfrak{n}\left(r, \frac{1}{\phi}\right)$. Hence (9) becomes

$$\begin{aligned} \mathfrak{T}(r, \phi) &< \frac{2}{3}\mathfrak{N}\left(r, \frac{1}{\phi}\right) + O(\log r + \log^+ \mathfrak{T}(r, \phi)) \\ &< \frac{2}{3}\mathfrak{T}(r, \phi) + O(\log r + \log^+ \mathfrak{T}(r, \phi)). \end{aligned}$$

Thus $3\mathfrak{T}(r, g) \leq \mathfrak{T}(r, \phi) + O(\log r) = O(\log r)$ initially outside a set of finite measure, and hence without exceptional set since $\mathfrak{T}(r, g)$ differs from a non-decreasing function by a bounded term [17] (see also [3, p.980]). \square

Lemma 14 *The function g has order at most 1.*

Proof. This proof is almost identical to [3, Lemma 3.2] and to corresponding arguments in [9]. Lemma 13 and an inequality of Levin-Ostrovskii [9, p.332] (see also [3, Lemma 2.2]) give

$$\int_R^\infty \frac{m_{0\pi}(r, g)}{r^3} dr \leq \int_R^\infty \frac{\mathfrak{m}(r, g)}{r^2} dr = O\left(\frac{\log R}{R}\right), \quad R \rightarrow \infty, \quad (10)$$

in which

$$m_{0\pi}(r, g) = \frac{1}{2\pi} \int_0^\pi \log^+ |g(re^{i\theta})| d\theta.$$

But g is real on the real axis and has finitely many poles and so

$$T(r, g) = m(r, g) + O(\log r) = 2m_{0\pi}(r, g) + O(\log r)$$

and (10) now gives, as $R \rightarrow \infty$,

$$\frac{T(R, g)}{R^2} \leq 2 \int_R^\infty \frac{T(r, g)}{r^3} dr \leq 4 \int_R^\infty \frac{m(r, g)}{r^2} dr + O\left(\frac{\log R}{R^2}\right) = O\left(\frac{\log R}{R}\right),$$

from which the lemma follows. \square

The rest of the proof is similar to the proof of Theorem 7. However, the arguments simplify considerably since we have restricted attention to the case $n = 2$ and since we do not have to count the number of non-real critical points as precisely as in the proof of Theorem 7.

We put

$$G(z) = z - \frac{1}{3}g(z)^3.$$

Then the zeros of G' are exactly the solutions of $g(z)^2 g'(z) = 1$, so all critical points of G , with finitely many exceptions, are real. By Lemma 14 and the Denjoy-Carleman-Ahlfors theorem, G has at most 2 finite asymptotic values. Thus the number of non-real singularities of G^{-1} is finite.

The fixed points of G are all multiple and they coincide with zeros of g . If g has finitely many zeros then $g(z) = p(z) \exp(az)$, where p is a rational function and $a \in \mathbf{R} \setminus \{0\}$. For such g , it is easy to see that the equation $g'(z)g(z)^2 = 1$ has infinitely many non-real solutions.

Hence we may assume that g has infinitely many zeros. A non-real zero of g is a non-real multiple fixed point of G , and Lemma 10 implies that the Leau domains associated to it contain a non-real singularity of G^{-1} . A multiple zero of g is a multiple fixed point of G of multiplicity at least 6, and Lemma 10 yields that at least 3 of the associated Leau domains contain a non-real singularity of G^{-1} . As the number of non-real singularities of G^{-1} is finite, we see that only finitely many zeros of g are non-real or multiple, and thus all but finitely many of the zeros of g are real and simple. This implies that there are infinitely many real zeros ζ of g with $g'(\zeta) < 0$. As in the proof of Theorem 7 we see that there are at least 2 non-real singularities of G^{-1} contained in the Leau domains associated to such a point ζ . Since the number of non-real singularities of G^{-1} is finite, we deduce that there are only finitely many real zeros ζ of g satisfying $g'(\zeta) < 0$, a contradiction. \square

5 Proof of Theorem 9

Let $m \geq 3$, let f , G and c be as in the hypotheses, and define g and H by

$$g = \frac{f' + c}{f^m}, \quad \frac{f' + c}{G} = \frac{g}{g + 1},$$

$$f'' - \left(\frac{G'}{G}\right)(f' + c) = \frac{g'G}{(g + 1)^2} = \frac{f^m g'}{g + 1} = f^m H. \quad (11)$$

The third equation in (11) is obtained by differentiating the second and multiplying by G . The function g is non-constant, poles of g are zeros of f , and $g(z) = -1$ implies $G(z) = 0$. We may assume that $H \not\equiv 0$, since $H \equiv 0$ implies that g is constant. Let $S(r, f)$ denote any quantity which is $o(T(r, f))$ as $r \rightarrow \infty$ outside a set of finite measure.

Lemma 15 *We have, as $r \rightarrow \infty$,*

$$(m - 1)m(r, f) = m(r, f^{m-1}) \leq m \left(r, \frac{1}{H}\right) - \alpha \log r + S(r, f), \quad (12)$$

in which: $\alpha = 0$ if f is transcendental; $\alpha = 2$ if $f(\infty) = \infty$; $\alpha = 2$ if $f(\infty) \in \mathbf{C} \setminus \{0\}$ and $g(\infty) \neq -1$; $\alpha = 1$ otherwise.

Proof. For transcendental f we apply the method of Clunie's lemma [7, Lemma 3.3, p.68]. Separating into the two cases $|f| < 1$, $|f| \geq 1$ and using (11) we obtain

$$|f^{m-1}H| \leq |H| + \left|\frac{f''}{f}\right| + \left|\frac{G'}{G}\right| \left(\left|\frac{f'}{f}\right| + |c|\right), \quad m(r, f^{m-1}H) = S(r, f),$$

and (12) follows on writing $f^{m-1} = (f^{m-1}H)/H$. Suppose now that f is rational. If f has a pole of multiplicity $p \in \mathbf{N}$ at ∞ , then g has a zero of multiplicity $p(m - 1) + 1$ and H a zero of multiplicity $p(m - 1) + 2$ at ∞ , which gives (12) with $\alpha = 2$. If $f(\infty) \in \mathbf{C}$ then $m(r, f) = O(1)$ and H has at least a simple zero at ∞ , while if $f(\infty) \in \mathbf{C} \setminus \{0\}$ and $g(\infty) \neq -1$ then the zero of H at ∞ is at least double. \square

From Jensen's formula and (12) we obtain at once, since $m(r, H) = S(r, f)$,

$$(m - 1)T(r, f) \leq (m - 1)N(r, f) - N \left(r, \frac{1}{H}\right) + N(r, H) - \alpha \log r + S(r, f). \quad (13)$$

Now if f has a pole in \mathbf{C} of multiplicity q then g has a zero of multiplicity $mq - q - 1 \geq 1$ and so H has a zero of multiplicity $mq - q - 2 = (m - 1)q - 2$. So the contribution to $(m - 1)n(r, f) - n(r, 1/H)$ from this pole is 2. A pole of H is simple, and must be a zero or pole of $g + 1$, and poles of g are zeros of f . Thus (13) yields

$$(m - 1)T(r, f) \leq 2\bar{N}(r, f) - N_0\left(r, \frac{g+1}{g'}\right) + \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}_1\left(r, \frac{1}{g+1}\right) - \alpha \log r + S(r, f). \quad (14)$$

Here $N_0\left(r, \frac{g+1}{g'}\right)$ counts zeros of $\frac{g'}{g+1}$ which are not poles of f , and $\bar{N}_1\left(r, \frac{1}{g+1}\right)$ counts zeros of $g + 1$ which are not zeros of f . Write

$$\bar{N}_1\left(r, \frac{1}{g+1}\right) = \bar{N}_{1,R}\left(r, \frac{1}{g+1}\right) + \bar{N}_{1,NR}\left(r, \frac{1}{g+1}\right),$$

in which the subscripts R, NR denote real, non-real zeros respectively.

Lemma 16 *We have, as $r \rightarrow \infty$,*

$$\bar{N}_{1,R}\left(r, \frac{1}{g+1}\right) \leq \bar{N}(r, f) + \bar{N}\left(r, \frac{1}{f}\right) + N_0\left(r, \frac{g+1}{g'}\right) + \beta \log r + S(r, f), \quad (15)$$

in which: $\beta = 0$ if f is transcendental; $\beta = -1$ if f is rational and $g(\infty) = -1$; $\beta = 1$ if f is rational and $g(\infty) \neq -1$.

Proof. Applying Rolle's theorem shows that between adjacent real zeros x_1, x_2 of $g + 1$ there must be at least one point x_0 which is a pole of g or a zero of g' but not a zero of $g + 1$. In the first case x_0 is a zero of f , while in the the second case x_0 either is a pole of f or contributes to $N_0\left(r, \frac{g+1}{g'}\right)$. If $g(\infty) = -1$ then the same argument may be applied on intervals of form $(-\infty, x_1), (x_2, \infty)$. \square

Combining (14) and (15) yields

$$(m - 1)T(r, f) \leq 3\bar{N}(r, f) + 2\bar{N}\left(r, \frac{1}{f}\right) + \bar{N}_{1,NR}\left(r, \frac{1}{g+1}\right) + (\beta - \alpha) \log r + S(r, f). \quad (16)$$

For transcendental f both assertions of the theorem follow at once from (16).

Suppose now that f is a rational function, and that all non-real zeros of $g + 1$ are zeros of f . If $f(\infty) = \infty$ then $\beta - \alpha \leq -1$ and a contradiction arises from (16) if $m \geq 6$ or if $m \geq 3$ and f is a polynomial. Next, if $f(\infty) = 0$ then $\beta - \alpha \leq 0$ and so (16) gives a contradiction for $m \geq 6$ since $\overline{N}\left(r, \frac{1}{f}\right) < (1 - \varepsilon)T(r, f)$ as $r \rightarrow \infty$, for some $\varepsilon > 0$. Finally, if $f(\infty) \in \mathbf{C} \setminus \{0\}$ then $\beta - \alpha \leq -1$ and again if $m \geq 6$ we obtain a contradiction from (16). \square

6 Examples

1. $f(z) = e^{-z} + 1$ gives $f' + f = 1$ so one cannot take $m = 1$ in Theorem 4.

2. $f(z) = -\tan z$ gives $f' + f^2 = -1$, so one cannot take $m = 2$ in Theorem 4.

3. $f(z) = 1/(2 \sin z)$ gives

$$f'(z) + f(z)^3 = \frac{1 - 2 \sin(2z)}{8 \sin^3 z}$$

and this has only real zeros, so Theorem 4 does not hold with $m = 3$.

$g(z) = 1/f(z) = 2 \sin z$ gives $g'(z)g(z) - 1 = 2 \sin 2z - 1$ which has only real zeros. So one cannot put $n = 1$ in Theorem 8.

4. For $f(z) = a - \tan a^4 z$ with $a = 12$ we obtain

$$w(t) = f' + f^4 = -a^4(1 + t^2) + (a + t)^4, \quad \text{where } t = -\tan a^4 z.$$

Indeed, $w(0) = 0$, $w'(0) > 0$ and $w(\pm\infty) = +\infty$. It follows that w has a negative root. Now $w(1) = -2 \times 12^4 + 13^4 = -12911 < 0$ which implies that w has 2 positive roots. Thus all 4 roots of w are real. The full preimage of the real line under $-\tan a^4 z$ is contained in the real line, so all roots of $f' + f^4$ are real. So Theorem 4 does not hold with $m = 4$.

Another example with this property is given by

$$f(z) = \frac{\sqrt{2} + \tan(4x)}{1 + \sqrt{2} \tan(4x)}$$

so that

$$f'(z) + f(z)^4 = -\frac{\tan^3(4x) (4\sqrt{2} + 7 \tan(4x))}{(1 + \sqrt{2} \tan(4x))^4}.$$

Taking $g = 1/f$ we see that Corollary 6 does not hold with $n = 2$.

5. For $f(z) = 1/z$ the function $f'(z) + f(z)^4 = (1 - z^2)/z^4$ has only real zeros, so the condition $m \geq 5$ in Theorem 3 cannot be weakened to $m \geq 4$ in order to conclude that $f' + f^m$ has non-real zeros. Equivalently $g(z) = 1/f(z) = z$ shows that in Corollary 5 the condition $n \geq 2$ does not yield the existence of non-real solutions of $g'g^n = 1$.

An example of degree 2 with these properties is given by

$$f(z) = \frac{5z + 3}{2z^2 + 8z + 3}$$

and $g(z) = 1/f(z)$. Here 0 is a zero of $g'g^2 - 1$ of multiplicity 3, and the other 3 zeros of this function are also real. This can be seen by numerical computation, but can also be deduced from Lemma 10, since $G(z) = z - g(z)^3/3$ has 4 Leau domains associated to the 2 real zeros of g , each of them containing a critical point, and symmetric about the real axis since $g' \geq 0$ on \mathbf{R} . If one contained a non-real critical point, then it would also contain the complex conjugate of this point, and this leads to a contradiction since there are only 4 distinct critical points.

6. The following examples shows that for transcendental meromorphic f one cannot take $m = 5$ or $m = 6$ in Theorem 9. Let $f(z) = -\tan z$. Then

$$f'(z) + f(z)^6 + 1 = (\tan z + 1)(\tan z - 1)(\tan^2 z + 1) \tan^2 z$$

has only real zeros.

For the case $m = 5$ let

$$5a^4 - 10a^2 + 1 = 0, \quad b = 5a - 10a^3, \quad c = -a^5 - b. \quad (17)$$

Set $f(z) = a + \tan(bz) = a + t$. Then

$$G = f' + f^5 + c = t(t^2 + 1)(t^2 + 5at + 10a^2 - 1).$$

Equation (17) has a root $a = \sqrt{1 - 2/\sqrt{5}} = 0.3249\dots$ for which we have $25a^2 - 4(10a^2 - 1) > 0$, so that all zeros of G are real.

The example

$$f(z) = -16z^2 + 8z + 2, \quad f'(z) + f(z)^2 - 12 = 256z^3(z - 1),$$

shows that Theorem 9 is sharp for polynomial f .

We suspect that Theorem 9 is not sharp for non-polynomial rational functions, nor for transcendental functions with finitely many poles.

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