Re-expansion Theorem. *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be a power series convergent in $|z| < R$*. Then for every* $a \in D(0, R)$ *there exists a power series, convergent in* $D(a, R - |a|)$*, such that*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \quad z \in D(a, R - |a|).$$

Proof. Choose arbitrary $r \in (|a|, R)$. Then we have convergent series with positive terms

$$\infty > \sum_{n=0}^{\infty} |a_n| r^n = \sum_{n=0}^{\infty} |a_n| \sum_{k=0}^{n} \binom{n}{k} (r - |a|)^k |a|^{n-k}.$$

Thus the double series

$$f(z) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} a_n (z-a)^k a^{n-k}$$

is absolutely convergent, and its terms can be regrouped according to the powers of $(z - a)$:

$$f(z) = \sum_{k=0}^{\infty} (z-a)^k \left(\sum_{n=k}^{\infty} \binom{n}{k} a^{n-k} \right),$$

where both series (inside and outside the parentheses) are convergent.

Thus the sum of a convergent power series is a regular function in its disc of convergence. Now we prove the converse:

Theorem. *Let* f *be an regular function in a disc* $D(a, R)$*. Then the Taylor series of* f *at the point* a *has radius of convergence at least* R*.*

Proof. Suppose that the theorem is not true and the radius of convergence is $r \in (0, R)$. According to the definition of a regular function it can be expanded into a Taylor series at every point z of the closed disc $|z| \leq (R+r)/2$.

Let $r(z)$ be the radius of convergence of this expansion. The re-expansion theorem implies that

$$r(z') \geq r(z) - |z - z'| \quad \text{and} \quad r(z) \geq r(z') - |z - z'|,$$

so $|r(z) - r(z') \leq |z - z'|$, thus $r(z)$ is a continuous function. as $r(z)$ is positive on the compact set $\{z : |z| < (R + r)/2\}$, it has a positive lower bound 2σ.

Let $M = \max_{|z| \leq r+\sigma} |f(z)|$. Then Cauchy's inequalities imply that

$$\frac{1}{n!}|f^{(n)}(z)| \leq M/\sigma^n, \quad |z| \leq r. \tag{1}$$

Differentiating the Taylor series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad |z| < r, \tag{2}$$

we obtain

$$\frac{1}{n!}|f^{(n)}(z)| = \sum_{n=k}^{\infty} \frac{k!}{n!(k-n)!} a_k z^{k-n}, \quad |z| < r$$

Applying Cauchy's inequality to the last series, and using (1), we obtain

$$\frac{k!}{n!(k-n)!}|a_k||z|^{k-n} \leq M/\sigma^n, \quad |z| < r, n \leq k, \quad |z| < r.$$

Passing to the limit, as $|z| \to r$, we can replace $|z|$ by r, so

$$\frac{k!}{n!(k-n)!}|a_k||r|^{k-n}\sigma^n \leq M.$$

adding these inequalities for $n = 0, \ldots, k$ and using the binomial formula, we obtain

$$|a_k|(r + \sigma)^k \leq M,$$

which implies that the radius of convergence of our series (2) is at least $r + \sigma$. This contradiction proves the theorem.