Stability in the Marcinkiewicz theorem

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Dedicated to the memory of I. V. Ostrovskii

Abstract

Ostrovskii's generalization of the Marcinkiewicz theorem implies that if an entire characteristic functions of a probability distribution satisfies $\log\log M(r,f) = o(r)$ and is zero-free then the distribution is normal. We show that under the same growth condition, absence of zeros in a wide vertical strip implies that the distribution is close to a normal one. This generalizes and simplifies a recent result of Michelen and Sahasrabudhe.

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Following Linnik [3], an entire function f is called a *ridge function* if $|f(z)| \leq |f(i\text{Im}\, z)|,\ z \in \mathbf{C}$. This definition is justified by Probability theory: characteristic functions of random variables are ridge functions when they are entire. We will apply the same name to subharmonic functions u in $\mathbf{C}$ satisfying

$$u(z) \leq u(i\text{Im}\, z), \quad z \in \mathbf{C}. \tag{1}$$

Classical theorem of Marcinkiewicz [5] says that all ridge entire functions of finite order without zeros are of the form $\exp(-az^2 + biz + c)$, where $a > 0$, b is real, and c is complex. This was generalized by Ostrovskii [7] who proved a conjecture of Linnik that the condition of finite order can be relaxed to

$$\log^+ \log |f(z)| = o(|z|), \quad z \to \infty.$$

This condition was further relaxed in [9] to

$$\liminf_{z\to\infty} \frac{\log^+ \log |f(z)|}{|z|} = 0. \tag{2}$$

Paper [8] contains a survey of further generalizations of Ostrovskii's result.

We prove a "stable version" of this theorem for entire functions which are free of zeros in vertical strips:

Theorem 1. *If u is a ridge subharmonic function in $\mathbf{C}$ satisfying*

$$\liminf_{r\to\infty} \frac{\log\max\{u(ir), u(-ir)\}}{r} = 0, \tag{3}$$

is harmonic in the strip

$$S(\Delta) = \{z : |\mathrm{Re}\, z| < \Delta\} \tag{4}$$

and normalized by $u(0) = u_x(0) = u_y(0) = 0$ and $u_{yy}(0) = 1$, then

$$|u(z) + \mathrm{Re}\,(z^2/2)| \leq c_0|z|^3/\Delta, \quad |z| \leq \Delta/3, \tag{5}$$

where c_0 is an absolute constant.

Example $u(z) = \cosh y \cos y - 1$ shows that the growth condition (3) is best possible. A new proof of Linnik's conjecture is obtained by setting $u = \log|f|$ and $\Delta = \infty$.

As a corollary we obtain a generalization of the recent theorem by Michelen and Sahasrabudhe [6, Thm. 4.1]:

Theorem 2. *Let X be a random variable with average μ and standard deviation σ. Suppose that the characteristic function f_X is entire, satisfies (2), and is free of zeros in the strip $\{z : |\mathrm{Re}\, z| < \delta\}$. Then the distribution function F_{X^*} of the random variable $X^* = (X - \mu)/\sigma$ satisfies*

$$|F_{X^*} - F_N|_\infty \leq \frac{c_1}{\sigma\delta},$$

where c_1 is an absolute constant, and N is the standard normal distribution with characteristic function $f_N(z) = \exp(-z^2/2)$.

This theorem was proved in [6] under the additional assumption that X takes values in the set $\{0, 1, \ldots, n\}$. We generalize the result and propose a shorter proof. We will use the

Phragmén–Lindelöf Theorem. *If a subharmonic function v in a strip S satisfies*

$$\liminf_{z\to\infty} \frac{\log^+ v(z)}{|z|} = 0, \tag{6}$$

and $v(z)\le 0,\ z\in\partial S$*, then* $v(z)\le 0$ *in* S*.*

Lemma 1. *If a harmonic function in a strip* $S(\Delta)$ *satisfies (3) and (1), then for all real* y*, the function* $x\mapsto u(x+iy)$ *is decreasing for* $x\in[0,\Delta/2]$*.*

Proof. Let us fix $s\in(0,\Delta/2)$ and let $z\mapsto z^*$ be the reflection with respect to the line $\mathrm{Re}\, z=s$, that is $z^*=2s-\overline{z}$. We define $u^*(z)=u(z^*)$, and

$$v(z)=\max\{u(z),u^*(z)\},\quad 0<\mathrm{Re}\, z<2s.$$

On the lines $\mathrm{Re}\, z=0$ and $\mathrm{Re}\, z=s$ we have $v(z)\le u(z)$. For a ridge function u, condition (3) implies (6) so u and v satisfy (6), and by the Phragmén–Lindelöf theorem we conclude that $v(z)\le u(z)$ in the strip $\{z:0<\mathrm{Re}\, z<s\}$. On the other hand $v(z)\ge u(z)$ by definition, so

$$v(z)=u(z),\quad 0<\mathrm{Re}\, z<s. \tag{7}$$

On the lines $\mathrm{Re}\, z=s$ and $\mathrm{Re}\, z=2s$ we have $v(z)\le u^*(z)$, so by a similar application of the the Phragmén–Lindelöf theorem we conclude that $v(z)\le u^*(z)$ in the strip $\{z:s<\mathrm{Re}\, z<2s\}$. On the other hand, $v(z)\ge u^*(z)$ by definition, so

$$v(z)=u^*(z),\quad s<\mathrm{Re}\, z<2s. \tag{8}$$

Since $v(z)$ is subharmonic, we have $v_x(s-0)\le v_x(s+0)$, and in view of (7), (8) we have

$$v_x(s-0)=u_x(s)\quad\text{and}\quad v_x(s+0)=u^*_x(s)=-u_x(s),$$

and so we obtain that $u_x(s)\le -u_x(s)$ that is $u_x(s)\le 0$, which proves the Lemma.

Lemma 2. *Let* Q *be the square,*

$$Q=\{x+iy:0<x<2,\ |y|<1\}, \tag{9}$$

and let $P(z,\zeta)$ *be the Poisson kernel of* Q*, where* $z=x+iy\in Q$*, and* $\zeta\in\partial Q$*. Then for* $\zeta\in\partial Q\backslash(-i,i)$ *we have*

$$P_x(0,\zeta)\ge c_2,$$

where c_2 *is an absolute constant.*

Lemma 3. *The family of harmonic functions in a vertical strip $S(\Delta)$ as in (4) satisfying (3), (1) and normalized both conditions*

$$u(0) = u_y(0) = 0, \quad u_{yy}(0) = 1,$$

is uniformly bounded from above on every compact set $K \subset S(\Delta/2)$ by a constant depending only on K and Δ.

Proof. By Lemma 1, harmonic functions $-u_x$ are positive in the right half of the strip, and $u_x(0, y) = 0$ in view of (1). Applying to them the Poisson representation in rectangles cQ where Q is defined in (9) and using Lemma 2, we obtain that the total measure in this representation is bounded. So u_x are uniformly bounded on compacts. We conclude that the analytic functions $u_x - iu_y$, are uniformly bounded on compacts. Since $u_x(0) = 0$ by the ridge property and $u_y(0) = 0$ by assumption, we conclude that functions u are uniformly bounded on compacts in $S(\Delta/2)$. This proves Lemma 3.

Proof of Theorem 1. We may assume without loss of generality that $\Delta \geq 1$. Consider the expansion at 0:

$$u(z) = \mathrm{Re}\left(-z^2/2 + \sum_{n=3}^{\infty} a_n z^n\right).$$

Let

$$u_\Delta = \Delta^{-2} u(z\Delta) = \mathrm{Re}\left(-z^2 + \sum_{n=3}^{\infty} a_n \Delta^{n-2} z^n\right), \quad z \in S(1).$$

By Lemma 3, its coefficients are uniformly bounded, therefore $|a_n| \leq c_3 \Delta^{2-n}$, and

$$\sum_{n=3}^{\infty} |a_n||z^n| \leq c_3 \Delta^{-1} \frac{|z|^3}{1 - |z|/\Delta} \leq c_0 |z|^3/\Delta, \quad \text{when} \quad |z| \leq \Delta/3.$$

This proves Theorem 1.

Derivation of Theorem 2 from Theorem 1. Following [6] and [2], we use the Berry–Esseen inequality

$$\sup_{t \in \mathbf{R}} |F_{X^*}(t) - F_Z(t)| \leq \frac{1}{\pi} \int_{-T}^{T} \left| \frac{f_{X*}(x) - e^{-x^2/2}}{x} \right| dx + \frac{c}{T}, \tag{10}$$

where c is an absolute constant.

This estimate can be found in [1, Ch. XVI, 3, Lemma 2] and in [4, Lemma 8.2.2].

We set $\Delta = \delta\sigma$. The statement of Theorem 2 is meaningful only when Δ is large, so we assume that $\Delta > c_0$, where c_0 is the constant in Theorem 1.

We are going to apply Theorem 1 to $u = \log|f_{X^*}|$, where f_{X*} is the characteristic function of X^*. Since X^* is normalized, u is normalized as required in Theorem 1. Since by assumption the characteristic function f_X has no zeros in the strip $S(\delta)$, the function f_{X^*} has no zeros in the strip $S(\Delta)$. Then Theorem 1 implies that

$$f_{X^*}(x) = \exp(-x^2/2 + R(x)), \quad \text{where} \quad |R(x)| \leq c_0|x|^3/\Delta, \quad |x| < \Delta/2.$$

Set $T = \Delta/(4c_0)$ in (10). To estimate the integral in (10) we break it into two parts:

Let

$$a := (\Delta/c_0)^{1/3} \geq 1.$$

When $|x| < a$, we have $|R(x)| \leq 1$, so $|e^{R(x)} - 1| \leq 2|R(x)| \leq 2c_0x^3/\Delta$, so

$$\int_{-a}^{a} \left| \frac{f_{X^*}(x) - e^{-x^2/2}}{x} \right| dx = \frac{2c_0}{\Delta} \int_{-\infty}^{\infty} e^{-x^2/2} x^2 dx \leq c_5/\Delta.$$

When $|x| \in [a, T]$ we use

$$f_{X^*}(x) = \exp(-x^2/2 + R(x))$$

and

$$x^2(-1/2 + |x|c_0/\Delta) \leq x^2(-1/2 + 1/4) = -x^2/4.$$

So

$$\int_{|x| \in [a,T]} \left| \frac{f_{X*}(x) - e^{-x^2/2}}{x} \right| dx \leq 4 \int_{a}^{\infty} e^{-x^2/4} dx \leq c_6/\Delta.$$

This completes the proof.

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References

[1] W. Feller, An introduction to probability theory and its applications, Vol. II, J. Wiley NY, 1971.

[2] J. Lebowitz, B. Pittel, D. Ruelle and E. Speer, Central limit theorems, Lee–Young zeros, and graph-counting polynomials, J. Comb. Theory, A 141 (2016) 147–183.

[3] Yu. Linnik, Decomposition of probability distributions, Dover Publ., New York, 1964.

[4] Yu. Linnik and I. Ostrovskii, Decompositions of random variables and vectors, AMS, Providence, RI, 1977.

[5] J. Marcinkiewicz, J. Sur une propriété de la loi de Gauss, Math. Z. 44 (1939), no. 1, 612—618.

[6] M. Michelen and J. Sahasrabudhe, Central limit theorems and the geometry of polynomials, arXiv:1908.009020v2.

[7] I. V. Ostrovskii, On the application of a law established by Wiman and Valiron to the investigation of characteristic functions of probability laws, Sov. Math., Dokl. 3, 433–437 (1962).

[8] A. Vishnyakova, I. Ostrovskii and A. Ulanovskii, On a conjecture of YU. V. Linnik, Leningrad Math. J., 2 (1991) N 4, 765–773.

[9] V. Zimogljad, The growth of entire functions that satisfy special inequalities, Teor. Funkcii Funkcional. Anal. i Prilozen., 6 (1968) 30—41.

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