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On the Riesz Charge of the Lower Envelope of δ-Subharmonic Functions

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Abstract. By potential theoretic methods involving the Cartan fine topology a recent result by two of the authors is extended as follows: The Riesz charge of the lower envelope of a family of 3 or more δ-subharmonic functions (no longer supposed continuous) in the plane equals the infimum of the charges of the lower envelopes of all pairs of functions from the family. As a key to this it is shown in two different ways that the (fine) harmonic measures of any 3 pairwise disjoint finely open planar sets have Borel supports with empty intersection. One proof of this uses the Jordan curve theorem and the fact that the set of inaccessible points of the fine boundary of a fine domain is Borel and has zero harmonic measure; the other involves Carleman–Tsuji type estimates together with a fine topology version of a recent result of P. Jones and T. Wolff on harmonic measure and Hausdorff dimension.

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Introduction

Let w be a δ-subharmonic function in a domain $\Omega \subset \mathbb{C}$, i.e., $w = u - v$, where the functions u and v are subharmonic in Ω. Then w is well-defined quasi-everywhere (q.e.) in Ω, that is: up to a polar set, in other words a set which locally has zero outer logarithmic capacity. The set of all δ-subharmonic functions in Ω is closed under the operation of taking the lower envelope (that is, the pointwise minimum) of finite sets of such functions. We denote this operation by $\wedge$. Let $\mu[w]$ denote the Riesz charge associated with a δ-subharmonic function w in Ω. There is a natural ordering on the set of all locally finite Borel charges (that is, signed Borel measures) on $\Omega: \mu \geqslant \nu$ if $\mu - \nu$ is a (nonnegative) measure. This relation induces the least upper bound $\vee\, \mu_j$ and the greatest lower bound $\wedge\, \mu_j$ of any finite set of charges $\{\mu_j\}$, and we write $|\mu| = \mu \vee 0 + (-\mu) \vee 0$. We follow the convention that $\mu[w] \geqslant 0$ corresponds to *subharmonic* functions w.

The main goal of this paper is to prove the following theorem (trivial for $m = 2$):

THEOREM 1. *Let $w_1, \ldots, w_m$ be δ-subharmonic functions in a domain $\Omega \subset \mathbb{C}, m \geqslant 3$. Then*

$$\mu\left[\bigwedge_{j=1}^{m} w_j\right] \geqslant \bigwedge_{1 \leqslant j < k \leqslant m} \mu[w_j \wedge w_k].$$

COROLLARY 1. *Let for any pair (j, k), $j \neq k$, the function $w_j \wedge w_k$ be subharmonic. Then the function $\wedge_{j=1}^{m} w_j$ is subharmonic too.*

Theorem 1 is very useful in the subharmonic approach to the value-distribution theory of meromorphic functions, cf. [7], [8]. Theorem 1 follows from Theorems 2 and 3 below, which appear to have independent interest. In [7] Theorem 1 and Corollary 2 were proved for continuous δ-subharmonic functions. For the general case, and for Theorems 2 and 3, we need some notions concerning the Cartan *fine topology* and related subjects, cf. [5], [9]. We do not assume that the reader is closely familiar with these notions and facts, and we gather them in Section 1. Concepts pertaining to the fine topology are marked 'fine(ly)'. If E is a subset of an open set $\Omega \subset \mathbb{C}$, the set of all fine limit points of E in Ω is called the *base* of E (relative to Ω) and denoted by $b(E)$.

In Theorem 2, Ω denotes a *Green set*, i.e., an open subset of $\mathbb{C}$ having a Green function. For any set $E \subset \Omega$ and any point $z \in \Omega$ we denote by ε_z^E the *swept-out* measure (relative to Ω) of the point-mass ε_z on E (see Section 1). If E is finely closed in Ω and if $z \in \Omega \backslash E$, then ε_z^E may be regarded as a (generalized) harmonic measure for the finely open set $\Omega \backslash E$ relative to Ω. When speaking of a set of measure 0 for a measure ν, the set will always be assumed to be ν-measurable.

THEOREM 2. *Let Ω be a Green set in $\mathbb{C}$; let w be a δ-subharmonic function in Ω; let $\mu = \mu[w]$ denote the Riesz charge of w; and let E denote the base of $\{z : w(z) = 0\}$ relative to Ω. Then $|\mu|(F) = 0$ for any set $F \subset E$ such that $\varepsilon_z^E(F) = 0$ for every $z \in \Omega \backslash E$, or equivalently that $\varepsilon_z^E(F) = 0$ for some point z in each fine connectivity component of $\Omega \backslash E$.*

Because ε_z^E is supported by the *fine boundary* $\partial_f E$ of E when $z \in \Omega \backslash E$, Theorem 2 implies, in particular, the known fact that $|\mu|(\mathrm{int}_f E) = 0$, where $\mathrm{int}_f E = E \backslash \partial_f E$ denotes the *fine interior* of E, cf. e.g. [5, p. 186].

When E is a subset of an open set $\Omega \subset \mathbb{C}$, a point $\zeta \in E$ is called *accessible* from $\Omega \backslash E$ if there exists a continuous map $p: [0, 1] \to \mathbb{C}$ such that $p(t) \in \Omega \backslash E$ for $0 \leqslant t < 1$, and $p(1) = \zeta$. This notion is clearly independent of Ω ($\supset E$). Taking for F in Theorem 2 the set E^* of points of E which are *not* accessible from $\Omega \backslash E$, we obtain in view of Lemma 3 in Section 2 the following corollary.

COROLLARY 2. *In the notation of Theorem 2, the set E^* of all points of E which are inaccessible from $\Omega \backslash E$ is a Borel set, and $|\mu|(E^*) = 0$.*

Unlike Theorem 2 itself, Corollary 2 remains valid if E is understood more generally to be any subset of Ω which differs from $\{z : w(z) = 0\}$ only by polar set, except that then E^* need no longer be Borel, but E^* will still be $|\mu|$-measurable with measure 0.

Like Theorem 1, Corollary 2 has a local nature, and so the assumption that Ω be a Green set can be dropped. Actually, Theorem 2 itself likewise remains valid for any open set $\Omega \subset \mathbb{C}$, when ε_z^E is replaced by the (generalized) harmonic measure $\omega(\Omega \backslash E, z)$ for the finely open set $\Omega \backslash E$ evaluated at $z \in \Omega \backslash E$, see Remark 1 in Section 2.

THEOREM 3. *Let D_1, D_2, D_3 be three pairwise disjoint, finely open subsets of a Green set $\Omega \subset \mathbb{C}$. The harmonic measures for D_1, D_2, D_3 have Borel supports S_1, S_2, S_3 such that $S_1 \cap S_2 \cap S_3 = \emptyset$.*

More precisely, there exist Borel sets S_1, S_2, $S_3 \subset \Omega$ such that $S_1 \cap S_2 \cap S_3 = \emptyset$ and that, for each $i = 1, 2, 3$, $\Omega \backslash S_i$ has measure 0 with respect to $\varepsilon_z^{\Omega \backslash D_i}$ for every $z \in D_i$. Theorem 3 remains valid when Ω is replaced by the entire plane $\mathbb{C}$ and $\varepsilon_z^{\Omega \backslash D_i}$ by the 'full' (generalized) harmonic measure $\omega(D_i, z)$, see Remark 1. Similarly as to Lemma 3 (Section 2) and Lemma 4 (Section 6).

In Sections 3, 4, and 5 we prove Theorems 2, 3, and 1, respectively. A second proof of Theorem 3 will be given in Section 6.

Theorem 2 and its corollary extend immediately to $\mathbb{R}^n$, $n \geqslant 3$, in place of $\mathbb{C}$, and Theorem 2 even extends to suitable harmonic spaces. We do not know whether Theorem 1, or equivalently Theorem 3 (cf. [6]), extends to $\mathbb{R}^n$, but they do not extend to harmonic spaces, as shown in Section 7.

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1. Preliminaries about the Fine Topology

The fine topology of classical potential theory in $\mathbb{C}$ is defined as the coarsest topology on $\mathbb{C}$ making every subharmonic function continuous. A well-known theorem of H. Cartan states that a set U is a fine neighborhood of a point $\zeta \in U$ if and only if $\complement U$ is *thin* (effilé) at ζ in the sense of Brelot, i.e., there should exist a subharmonic function v in a neighborhood of ζ such that

$$v(\zeta) > \limsup_{z \to \zeta, z \in \complement U} v(z).$$

Some connectivity properties of the fine topology will be recalled in the proof of Lemma 2 in Section 2.

We shall now consider potential theoretic notions relative to a fixed open set $\Omega \subset \mathbb{C}$. Topological and potential theoretic notions will then be understood relative to Ω. For instance, the *base* $b(E)$ of a set $E \subset \Omega$ is the set of all points of Ω at which E is not thin. Equivalently, $b(E)$ is the set of all fine limit points of E in Ω, and $b(E)$ is therefore finely closed (relative to Ω). Moreover, $b(E)$ is a (usual) G_δ set; $E \setminus b(E)$ is polar; and $b(b(E)) = b(E)$. Cf., e.g., [5, p. 177].

A set $e \subset \Omega$ is polar if and only if $b(e) = \emptyset$ (i.e., e has only finely isolated points). A polar set is finely closed. Two subsets of Ω which differ only by a polar set have the same base. Note that, when w is δ-subharmonic in an open set $\Omega \subset \mathbb{C}$, the set $E = \{z : w(z) = 0\}$ differs only by a polar set from its fine closure F or from its base $b(E) = b(F)$.

Now suppose that Ω is a *Green set*, and let G be the Green kernel for Ω. For any superharmonic function $s \geqslant 0$ in Ω and any set $E \subset \Omega$ the function R_s^E defined for $z \in \Omega$ by

$$R_s^E(z) = \inf\{u(z) : u \text{ superharmonic} \geqslant 0 \text{ in } \Omega,\ u \geqslant s \text{ in } E\}$$

is called the reduced function of s, and its lower semicontinuous regularization $\hat{R}_s^E$ is called the *swept-out* function of s (relative to Ω). When s is the Green potential $G\mu$ of a measure μ on Ω, the swept-out function $\hat{R}_{G\mu}^E$ is the Green potential of a certain measure on Ω denoted by μ^E and called the swept-out of μ on E (relative to Ω). When sweeping a measure on any two subsets of Ω which differ only by a polar set, the swept-out measures are the same.

Let ε_z be the Dirac measure at a point $z \in \Omega$. The following identity holds for any measure μ on Ω such that $G\mu$ is superharmonic in Ω, any set $E \subset \Omega$, and any Borel set $B \subset \Omega$:

$$\mu^E(B) = \int_\Omega \varepsilon_z^E(B)\mu(\mathrm{d}z) \tag{1.1}$$

(cf. [5, p. 160]). By linearity one can extend the definition of the sweeping operation so as to apply to charges μ on Ω (always such that $G|\mu|$ is superharmonic, i.e., $G|\mu| \not\equiv \infty$ in every component of Ω).

If D is a finely open set and $E = \Omega \setminus D$, the *generalized Green function* for D with pole at z is

$$G^D(\cdot, z) = G(\cdot, z) - G\varepsilon_z^E = G\varepsilon_z - \hat{R}_{G\varepsilon_z}^E. \tag{1.2}$$

This function $G^D(\cdot, z)$ is non-negative, subharmonic in $\Omega \setminus \{z\}$, and its Riesz measure (on $\Omega \setminus \{z\}$) is the harmonic measure ε_z^E for D relative to Ω. We have $G^D(\cdot, z) = 0$ quasi-everywhere in E (and everywhere in E if E is a base). Further properties of $G^D(\cdot, z)$ considered in D are given in [10].

REMARK 1. The 'full' (generalized) harmonic measure for a finely open set $D \subset \mathbb{C}$ evaluated at a point $z \in D$ may be defined as a measure on $\mathbb{C}$ as follows. If $\complement D$ is polar, we set $\omega(D, z) = 0$. Now suppose that $\complement D$ is not polar (hence not even inner polar, by Choquet's capacitability theorem). By Myrberg's theorem there then exist Green sets $\Omega \supset D$ in $\mathbb{C}$. It can be shown that the (generalized) Green function $G^D(\cdot, z)$, defined in D relatively to such a Green set Ω by (1.2) above, is independent of the choice of Ω. Moreover, $G^D(\cdot, z)$ has a unique extension to all of $\mathbb{C}$ such that $G^D(\cdot, z)$ is subharmonic in $\mathbb{C}\backslash\{z\}$ and equal to 0 q.e. in $\mathbb{C}\backslash D$ (and more precisely $G^D(\cdot, z) = 0$ everywhere in $b(\mathbb{C}\backslash D)$). The Riesz measure of this subharmonic extension of $G^D(\cdot, z)$ from $D\backslash\{z\}$ to $\mathbb{C}\backslash\{z\}$ will be called simply the *harmonic measure* for D at z, and we denote it by $\omega(D, z)$. The measure $\omega(D, z)$ is carried by the fine boundary $\partial_f D$ of D, and even by the fine boundary of the fine component of z in D. It can be shown that $\omega(D, z)$ has total mass 1 and does not charge the polar sets. The trace of $\omega(D, z)$ on any Green set $\Omega \supset D$ in $\mathbb{C}$ is $\varepsilon_z^{\Omega\backslash D}$ (sweeping relative to Ω). In case of, say, a *bounded*, finely open set $D \subset \mathbb{C}$ we therefore have $\omega(D, z) = \varepsilon_z^{\Omega\backslash D}$ for any $z \in D$ and any Green set Ω in $\mathbb{C}$ containing the fine closure of D in $\mathbb{C}$. For a *usual* open set $D \subset \mathbb{C}$, $\omega(D, z)$ equals the usual harmonic measure for D at z. – An alternative characterization of the harmonic measure $\omega(D, z)$ for a finely open set D of non-polar complement is obtained by solving the *fine Dirichlet problem* for a bounded and finely continuous function ϕ on $\partial_f D$. The solution H_ϕ^D is the unique bounded, *finely harmonic* function in D which has the fine limit $\phi(y)$ q.e. for $y \in \partial_f D$ (namely for every $y \in (\partial_f D) \cap b(\complement D)$). It can be shown that

$$H_\phi^D(z) = \int \phi \; d\omega(D, z), \quad z \in D.$$

In the case where D is bounded, this is contained in [9, §14]. – In the rest of the paper we shall not draw on the present remark since most of its content has not been published, and we shall therefore again refer to suitable Green sets in $\mathbb{C}$; this will cause no real loss of generality.

2. Auxiliary Results

For the proof of Theorem 2 we need the following lemma.

LEMMA 1. *For $i = 1, 2$ let Ω_i be a Green set in $\mathbb{C}$, let $E_i \subset \Omega_i$ be finely closed relatively to Ω_i, and let Ω_0 be an open subset of $\Omega_1 \cap \Omega_2$ such that*

$$E_1 \cap \Omega_0 = E_2 \cap \Omega_0.$$

For any Borel set $B \subset \Omega_0$ we then have the bi-implication

$${}^{\Omega_1}\varepsilon_z^{E_1}(B) = 0 \textit{ for all } z \in \Omega_1\backslash E_1 \Leftrightarrow {}^{\Omega_2}\varepsilon_z^{E_2}(B) = 0 \textit{ for all } z \in \Omega_2\backslash E_2.$$

Here the left superscript Ω_i, $i = 1, 2$, indicates sweeping relative to Ω_i. Using results about the fine Dirichlet problem [9, §14], this lemma can be proved in a similar way as in the well-known special case where E_i is relatively closed in Ω_i, $i = 1, 2$, in the usual topology. We omit the details because Lemma 1 only enters effectively in the reduction of Theorem 2 to the case where w is the Green potential of a charge, and Theorem 1 easily reduces to the case of such potentials (even equal to 0 outside some disk, if we like).

In our first proof of Theorem 3 we use the following consequence of the famous puzzle about three houses and three wells.

LEMMA 2. *Let D_1, D_2, D_3 be pairwise disjoint, finely open sets in $\mathbb{C}$. Then the set X of points of $\mathbb{C}$ accessible from D_1, D_2, D_3 simultaneously is at most countable.*

In fact, if D_1, D_2, D_3 are moreover finely connected, or equivalently pathwise (even polygonally) connected [12, p. 113], then there are at most 2 points simultaneously accessible from these three fine domains. This is nothing but Euler's puzzle – an easy consequence of the Jordan curve theorem, cf. Whyburn [18, p. 316] or [7]. The general case follows because the fine topology is locally connected, cf. [9, p. 92], and a finely open set has at most countably many fine components; furthermore, a path in $\mathbb{C}$ contained in a finely open set D is contained in one of the fine components of D (because the union of those fine components of D which meet the path is pathwise connected, hence also finely connected in the present, planar case).

The following lemma enters in the proof of Corollary 2 and in the first proof of Theorem 3.

LEMMA 3. *Let Ω be a Green set in $\mathbb{C}$, and let E be a relatively finely closed subset of Ω. The set E^* of all points of E which are inaccessible from $\Omega\backslash E$ is then a Borel set, and*

$$\varepsilon_z^E(E^*) = 0, \quad z \in \Omega\backslash E.$$

Proof. To prove that E^* is a Borel set (in the usual sense) we adapt the proof given by Mazurkiewicz [16] for the case of a usual compact set $E \subset \mathbb{C}$. For this adaptation we use once more the connectivity properties of the fine topology on $\mathbb{C}$ mentioned after Lemma 2 above.

The open set $\Omega \subset \mathbb{C}$ is a Polish space in the sense of [2, §6]. Accordingly, let ρ denote a metric on Ω inducing the standard topology on Ω and such that the metric space (Ω, ρ) is complete, necessarily with a countable base $(\Omega_n)_{n\in\mathbb{N}}$ of open sets. Replacing ρ by $\min\{1, \rho\}$, we may arrange that $\rho \leqslant 1$.

Following [16] we introduce on $D := \Omega\backslash E$ a new metric ρ^* defined by

$$\rho^*(x, y) = \inf\{\operatorname{diam} L : L \in \mathscr{L}(x, y)\}, \quad x, y \in D,$$

where $\mathscr{L}(x, y)$ denotes the set of all (ranges of continuous) paths L in (D, ρ) leading from x to y, and diam L denotes the diameter of L with respect to ρ. (If $\mathscr{L}(x, y) = \varnothing$, define $\rho^*(x, y) = 1$.)

Let $\hat{D} = (\hat{D}, \rho^*)$ denote the completion of (D, ρ^*). The complete metric space $(\hat{D}, \rho^*)$ is a Polish space because it is separable. In fact, a countable dense subset of (D, ρ^*) and hence of $(\hat{D}, \rho^*)$ can be obtained as $X = \bigcup_{n \in \mathbb{N}} X_n$ where, for each $n \in \mathbb{N}$, X_n consists of one point from each fine component of the finely open set $D \cap \Omega_n$. This uses the pathwise connectedness of fine domains in $\mathbb{C}$.

The identity map $(D, \rho^*) \to (D, \rho) \subset (\Omega, \rho)$ is contractive and hence extends uniquely to a contraction

$$\phi : (\hat{D}, \rho^*) \to (\Omega, \rho).$$

It is easy to see that every point of $\hat{D} \backslash D$ is accessible from D (in the topology defined by ρ^*), and further that

$$\phi(\hat{D}) = \Omega \backslash E^*,$$

which shows that E^* is the complement within Ω of the Souslin set $\phi(\hat{D})$. In particular, E^* is universally measurable, and this is really all that matters in the applications of Lemma 3 in the present paper.

To show that $\phi(\hat{D})$ and hence E^* are even Borel sets, one proceeds like in [16]. The main step is to show that the set Z of points $z \in \rho(\hat{D})$ for which the pre-image $\phi^{-1}(z)$ has 3 or more points is at most countable; for this one uses Lemma 2 above and the fine connectivity properties listed after it, cf. the analogous step in [16]. Next, $\hat{D} \backslash \phi^{-1}(Z)$ is a G_δ in the Polish space $(\hat{D}, \rho^*)$, hence itself a Polish space. Because ϕ is at most countable-to-one (even 2-to-1) on $\hat{D} \backslash \phi^{-1}(Z)$, the image $\phi(\hat{D}) \backslash Z$ is not only analytic, but a Borel set in Ω, and so are therefore $\phi(\hat{D})$ and $E^* = \Omega \backslash \phi(\hat{D})$, by a well-known theorem, see e.g. [3, Theorem 5.17].

For the last assertion of Lemma 3 we use that, for any Borel set $B \subset E$ and any point $z \in D$ $(= \Omega \backslash E)$, $\varepsilon_z^E(B)$ equals the probability that a Brownian particle in Ω, starting at z, eventually hits E, and finds itself in B the first time it hits E, cf. [1, p. 264]. When applied to $B = E^*$ (the points of which are never hit) this shows that indeed $\varepsilon_z^E(E^*) = 0$.

REMARK 2. Lemma 3 extends to $\mathbb{R}^n$ (in place of $\mathbb{C}$) except that E^* need then no longer be a Borel set (according to known examples with E a usual closed set in $\mathbb{R}^3$), but E^* is still the complement within Ω of the Souslin set $\phi(\hat{D})$, and in particular E^* remains universally measurable. This appears from the first part of the above proof.

3. Proof of Theorem 2

Write $D = \Omega \backslash E$ and let D_n denote the nth fine connectivity component of the finely open set D. Suppose that $F \subset E$ and that, for each n in question, $\varepsilon^E_{z_n}(F) = 0$ for some $z_n \in D_n$. Then $\varepsilon^E_z(F) = 0$ for every $z \in D$, see [9, p. 150]. In proving that $|\mu|(F) = 0$ we may assume that F is contained in a given disk Ω_0 with closure in Ω. Choose a Borel set A_n so that $F \subset A_n \subset E \cap \Omega_0$ and $\varepsilon^E_{z_n}(A_n) = 0$, and write $A = \bigcap_n A_n$. Then $F \subset A \subset E \cap \Omega_0$, and it remains to prove that $\mu(B) = 0$ for every Borel set $B \subset A$. Clearly, $\varepsilon^E_{z_n}(B) = 0$ and hence, as above,

$$\varepsilon^E_z(B) = 0 \quad \text{for every } z \in D. \tag{3.1}$$

For the proof that $\mu(B) = 0$ we consider first the case where $w = -G\mu$ for some charge μ on Ω (with $G|\mu|$ superharmonic); then $\mu[w] = \mu$. It is known that ε^E_z is supported by $b(E) = E$; and that ε^E_z does not charge the polar sets, except if $z \in E$, in which case $\varepsilon^E_z = \varepsilon_z$; cf. [5, p. 161, 183]. Because $G\mu = 0$ q.e. in E we therefore have

$$G\mu^E(z) = \int G\mu \, \mathrm{d}\varepsilon^E_z = 0 \quad \text{q.e. for } z \in \Omega,$$

cf. [5, p. 160]. It follows that

$$\mu^E = 0. \tag{3.2}$$

We now decompose

$$\mu = \lambda + \nu, \tag{3.3}$$

where $\lambda = \mu|_E$, $\nu = \mu|_D$. Then $\lambda^E = \lambda$ since λ is supported by $E = b(E)$, cf. [5, p. 183]. In view of (3.2), sweeping on E therefore leads to

$$\mu^E = \lambda + \nu^E = 0. \tag{3.4}$$

Using (1.1) and (3.1) we obtain

$$\nu^E(B) = \int_D \varepsilon^E_z(B) \mu(\mathrm{d}z) = 0.$$

By (3.4) we thus get $\lambda(B) = 0$, and we conclude from (3.3) that

$$\mu(B) = \nu(B) = 0$$

because $B \subset A \subset E$ and ν is supported by D.

In case of a general δ-subharmonic function w in Ω we choose a disk Ω_1 so that $\bar{\Omega}_0 \subset \Omega_1$, $\bar{\Omega}_1 \subset \Omega$. Writing $w = u - v$ with u, v superharmonic in Ω, hence lower bounded in Ω_1, we may assume that $u, v > 0$ in Ω_1. Now replace Ω by Ω_1 and u, v by their swept-out functions on Ω_0 relative to Ω_1, which are Green potentials of measures on Ω_1. These substitutions cause no change in the restrictions of u, v, and w to Ω_0, and therefore no change in the trace of $\mu = \mu[w]$ on Ω_0. It follows that $E \cap \Omega_0$ ($\supset B$) and $\mu(B)$ do not change. Moreover, we infer from Lemma 1 (taking $\Omega_2 = \Omega$, $E_2 = E$, $E_1 = E \cap \Omega_1$ there) that the Borel subsets of Ω_0 of ε_z^E-measure 0 for all $z \in \Omega \backslash E$ do not change, and so ${}^{\Omega_1}\varepsilon_z^{E_1}(B) = 0$ for every $z \in \Omega_1 \backslash E_1$. Consequently, $\mu(B) = 0$ by the above proof for the case where $w = -G\mu$.

4. First Proof of Theorem 3

For brevity write $E_i = b(\Omega \backslash D_i)$ (relative to Ω), and denote by $E_i^a = E_i \backslash E_i^*$ the set of points of E_i which are *accessible* from $\complement E_i$ (or equivalently from $\Omega \backslash E_i$). By Lemma 3, E_i^a is a Borel set because the base E_i is a Borel set. The finely open sets

$$\tilde{D}_i := \Omega \backslash E_i, \quad i = 1, 2, 3,$$

are pairwise disjoint because $b(A_1 \cup A_2) = b(A_1) \cup b(A_2)$ for any two sets $A_1, A_2 \subset \Omega$. When applied to these three sets $\tilde{D}_i$, Lemma 2 shows that

$$X := E_1^a \cap E_2^a \cap E_3^a$$

is countable, in particular polar. Now take

$$S_i = E_i^a \backslash X,$$

that is, the set of all points of $E_i = \Omega \backslash \tilde{D}_i$ which are accessible from $\tilde{D}_i$, but not from all three sets $\tilde{D}_1$, $\tilde{D}_2$, $\tilde{D}_3$ simultaneously. When $z \in \tilde{D}_i$, in particular when $z \in D_i$, the measure $\varepsilon_z^{E_i} = \varepsilon_z^{\Omega \backslash D_i}$ does not charge the polar sets, and it is supported by the Borel set E_i^a according to Lemma 3, and consequently also by S_i, which is likewise Borel.

Theorem 3 is proved. A different proof (without the use of accessible points and Brownian motion) will be given in Section 6 below.

5. Proof of Theorem 1

Because the theorem has a local nature we may suppose that Ω is a Green domain, e.g., a disk (and also, if we like, that each w_i is the Green potential of a charge). We begin with the case $m = 3$ and write

$$w_i^* = \bigwedge_{j \neq i} w_j, \quad w = \bigwedge_{1 \leqslant j \leqslant 3} w_j,$$
$$E_i = b(\{z : w_i(z) \geqslant w_i^*(z)\}), \quad D_i = \Omega \backslash E_i; \quad i = 1, 2, 3,$$

the bases being taken relative to Ω. The sets D_i are finely open, usual F_σ sets, and they are pairwise disjoint, cf. the beginning of the preceding proof. Clearly, $\{z : w_i(z) \geqslant w_i^*(z)\} = \{z : w(z) = w_i^*(z)\}$, and so

$$E_i = b(\{z : w(z) = w_i^*(z)\}). \tag{5.1}$$

With the above notations the conclusion of Theorem 1 reads

$$\mu[w](B) \geqslant \Big(\bigwedge_{1 \leqslant j \leqslant 3} \mu[w_j^*]\Big)(B) \tag{5.2}$$

for every Borel set $B \subset \Omega$. Let S_1, S_2, S_3 be Borel sets with the properties stated in Theorem 3. Writing $F_i = \Omega \backslash S_i$ we decompose $\Omega = F_1 \cup F_2 \cup F_3$ into 3 pairwise disjoint Borel sets $F_i' \subset F_i$. Then $\varepsilon_z^{E_i}(F_i') = 0$ for $z \in D_i$, $i = 1, 2, 3$. The given Borel set B decomposes into its parts in the 6 pairwise disjoint Borel sets $E_i \cap F_i'$ and $D_i \cap F_i'$; and it suffices therefore to prove (5.2) for any Borel set B contained in one of these 6 sets. We do this by applying Theorem 2 to the function $w - w_i^*$ and the base E_i, $i = 1, 2, 3$, cf. (5.1). If $B \subset E_i \cap F_i'$ for some i then $\varepsilon_z^{E_i}(B) = 0$ for $z \in D_i$, and hence

$$\mu[w](B) = \mu[w_i^*](B),$$

from which (5.2) follows. Suppose instead that for example $B \subset D_1 \cap F_1'$. Then B is contained in the fine interior of E_2, because D_1 is finely open and contained in $\Omega \backslash D_2 = E_2$. As explained after Theorem 2 it follows that $\mu[w](B) = \mu[w_2^*](B)$, and (5.2) ensues.

This establishes Theorem 1 in the case $m = 3$. The general case $m \geqslant 3$ follows by induction:

$$\begin{aligned} \mu[w_1 \wedge \dots \wedge w_m] &= \mu[(w_1 \wedge \dots \wedge w_{m-2}) \wedge w_{m-1} \wedge w_m] \\ &\geqslant \mu[w_1 \wedge \dots \wedge w_{m-2} \wedge w_{m-1}] \wedge \mu[w_1 \wedge \dots \wedge w_{m-2} \wedge w_m] \\ &\quad \wedge \mu[w_{m-1} \wedge w_m] \geqslant \bigwedge_{1 \leqslant j < k \leqslant m} \mu[w_j \wedge w_k]. \end{aligned}$$

6. Second Proof of Theorem 3

We may suppose that $\Omega \backslash D_j$ is a base. (Otherwise replace D_j by $\tilde{D}_j = \Omega \backslash b(\Omega \backslash D_j)$, and note that the finely open sets $\tilde{D}_1$, $\tilde{D}_2$, $\tilde{D}_3$ are again pairwise disjoint, as in the first proof of Theorem 3; and $\varepsilon_z^{\Omega \backslash \tilde{D}_j} = \varepsilon_z^{\Omega \backslash D_j}$.) Recall that D_j has at most countably many fine connectivity components, and if C_j denotes any one of them then $\varepsilon_z^{\Omega \backslash D_j} = \varepsilon_z^{\Omega \backslash C_j}$ for every $z \in C_j$, [9, p. 155]. In view of these circumstances it is enough to prove Theorem 3 in the case where D_1, D_2, D_3 are *finely connected*. Also recall that the sets of measure 0 with respect to $\varepsilon_z^{\Omega \backslash D_j}$ are then the same for all $z \in D_j$, [9, p. 150].

Fix arbitrary points $\zeta_j \in D_j$ and put $\mu_j = \varepsilon_{\zeta_j}^{\Omega \backslash D_j}$ (sweeping relative to Ω). It is sufficient to prove that the measures μ_j have supports with empty intersection, and this is equivalent to

$$\nu := \mu_1 \wedge \mu_2 \wedge \mu_3 = 0. \tag{6.1}$$

Let us denote by $u_j = G^{D_j}(\cdot, \zeta_j)$ the generalized Green function for D_j with pole at ζ_j, cf. (1.2). Then u_j is a non-negative function, subharmonic in $\Omega\backslash\{\zeta_j\}$, and its Riesz measure is μ_j. Furthermore, $u_j = 0$ in E_j.

For $z_0 \in \mathbb{C}$ and $t > 0$ we denote by $\theta_j(z_0, t)$ the Lebesgue measure of the set $\{\theta \in [0, 2\pi): z_0 + te^{i\theta} \in D_j\}$. We have

$$\theta_1(z_0, t) + \theta_2(z_0, t) + \theta_3(z_0, t) \leqslant 2\pi. \tag{6.2}$$

Recall that μ_j is supported by the *fine boundary* $\partial_f D_j$, cf. [5, p. 186]. Fix $z_0 \in \bigcap_j \partial_f D_j$ and $r_0 > 0$ so that $r_0 < \min_j |z_0 - \zeta_j|$, and write

$$\Theta_j(z_0, r) = \int_r^{r_0} \frac{dt}{t\theta_j(z_0, t)}, \quad r < r_0.$$

Each circle $|z - z_0| = t$ of radius $t \leqslant r_0$ meets $\mathbb{D}_j$ for every $j = 1, 2, 3$. (In fact, $\mathbb{D}_j \supset D_k$ for $k \neq j$, and D_k is finely connected, in particular connected in the usual topology; and because $z_0 \in \partial_f D_k \subset \partial D_k$, D_k contains points z with $|z - z_0| < t$ as well as the point ζ_k with $|\zeta_k - z_0| > r_0 \geqslant t$.) In view of this the well-known estimate of Carleman (cf. [14] combined with the proof of a related estimate of harmonic measure due to Tsuji [17, Theorem III.67 and Corollary, p. 116]) implies

$$u_j\left(z_0 + \frac{r}{2}e^{i\theta}\right) \leqslant C\exp(-\pi\Theta_j(z_0, r)), \tag{6.3}$$

where C is independent of $r < r_0$ and θ. (An alternative proof of (6.3) is described in Remark 3 below.)

Write $D(a, r) = \{z: |z - a| \leqslant r\}$. Combining (6.3) with the Jensen inequality we get for $r < r_0$

$$\begin{aligned}
\mu_j\left(D\left(z_0, \frac{r}{2e}\right)\right) &\leqslant \int_0^{r/2} \mu_j(D(z_0, t))\frac{dt}{t} \\
&\leqslant \frac{1}{2\pi}\int_0^{2\pi} u_j\left(z_0 + \frac{r}{2}e^{i\theta}\right)d\theta \\
&\leqslant C\exp(-\pi\Theta_j(z_0, r)).
\end{aligned} \tag{6.4}$$

Using the Schwarz inequality and (6.2) we obtain

$$\begin{aligned}
3(\ln r_0 - \ln r)^2 &= \sum_{j=1}^{3}\left(\int_r^{r_0} \frac{\sqrt{\theta_j(z_0, t)}}{\sqrt{\theta_j(z_0, t)}}\frac{dt}{t}\right)^2 \\
&\leqslant \sum_{j=1}^{3} \int_r^{r_0} \theta_j(z_0, t)\frac{dt}{t}\int_r^{r_0} \frac{dt}{t\theta_j(z_0, t)} \\
&\leqslant 2\pi(\ln r_0 - \ln r)\max_j \Theta_j(z_0, r),
\end{aligned}$$

so

$$\max_j \Theta_j(z_0, r) \geq \frac{3}{2\pi}(\ln r_0 - \ln r).$$

Together with (6.4) this implies

$$\nu(D(z_0, r)) \leq \min_j \mu_j(D(z_0, r)) = O(r^{3/2}), \quad r \to 0. \tag{6.5}$$

Now we use the following

LEMMA 4. *For an arbitrary finely open subset D of a Green set $\Omega \subset \mathbb{C}$ the measure $\mu = \varepsilon_\zeta^{\Omega \setminus D}$, $\zeta \in D$, has a Borel support of Hausdorff dimension at most* 1.

For usual domains this result is due to P. Jones and T. Wolff [15]. Let us derive the required generalization. Denote by $u = G^D(\cdot, \zeta)$ the generalized Green function for D with pole at ζ, cf. (1.2). Write $E = \Omega \setminus D$, and let $F_n \subset E$ be a sequence of compacts such that $\Sigma_n \mu(F_n) = \mu(E)$ $(= \mu(\Omega))$. Consider the Green function u_n for the open set $\Omega \setminus F_n$ with the same pole ζ as u. We have $u_n \geq u$, and $u_n = u$ quasi-everywhere on F_n, so by a theorem of Grishin [13] the Riesz measure of u_n majorizes the Riesz measure of u on F_n. By the Jones–Wolff theorem the Riesz measure of u_n has a Borel support of Hausdorff dimension ≤ 1. It follows that the restriction of μ to each F_n has a Borel support of Hausdorff dimension ≤ 1, and so has therefore μ itself.

Let us now complete the proof of the theorem. By Lemma 4 the measure ν defined in (6.1) has a Borel support $S \subset \bigcap_j \partial_f D_j$ such that $\dim S \leq 1$. Also ν satisfies (6.5) at each point $z_0 \in S$. If $\nu(S) > 0$ then by Egorov's theorem (applied e.g. to the sequence of functions $(r_n)^{-5/4}\nu(D(z, r_n))$ on S, with $r_n = 2^{-n}r_0$) there exist a set $S' \subset S$ with $\nu(S') > \nu(S)/2$ and a number $R > 0$ such that

$$\nu(D(z, r)) \leq r^{5/4}, \quad r < R, \quad z \in S'. \tag{6.6}$$

It follows from the definition of the Hausdorff dimension that, given any $\delta > 0$, there exists a covering of S' by disks Δ_n of radii $r_n < R$ such that $\Sigma r_n^{5/4} < \delta$. We may suppose that the centers of these disks lie on S'. So by (6.6)

$$\tfrac{1}{2}\nu(S) < \nu(S') \leq \sum_n \nu(\Delta_n) \leq \sum_n r_n^{5/4} < \delta,$$

and it follows that $\nu(S) = 0$, which proves (6.1) and the theorem.

REMARK 3. Alternatively, the proof of (6.3) outlined above could be replaced by an extension of the quoted corollary in [17, p. 116] to the case of *fine domains* $D \subset \Omega$. (D in [17] corresponds to D_j in the above proof.) This extension is based on an approximation theorem due to Choquet [4, p. 91] applied to the set $X = \Omega \setminus D$, which is thin at each point of the fine domain D. Thus there exists a decreasing sequence of usual domains $D^{(n)}$ with $D \subset D^{(n)} \subset \Omega$ such that the capacity of $D^{(n)} \setminus D$ with respect to the Green kernel G for Ω tends to 0 as $n \to \infty$. It follows that the two sets $\bigcap_n D^{(n)}$ and D differ only by a polar set and hence have the same intersection with the circle $|z| = t$ for almost every t. (The center 0 of these circles corresponds to z_0 above.) It remains only to apply the monotone convergence theorem in order to prove the desired extension of the estimate of harmonic measure given in [17, p. 116] to the case of a fine domain D, noting that the relevant evaluation of harmonic measure (denoted by $u_r(z)$ in [17]) does not decrease when D is replaced by $D^{(n)} \supset D$. Finally, (6.3) above is a straightforward consequence of this estimate, applied to $D = D_j$.

7. An Example from Axiomatic Potential Theory

We give an example of a Brelot harmonic space satisfying the axiom of domination and having a Green function, and such that Theorems 1 and 3 do not hold. The example is due to A. Cornea (unpublished) who devised it for a different purpose, see [11, p. 184 f.]. In this example all subharmonic functions are finite and continuous, and so the fine topology coincides with the initial topology, and no point forms a polar set. There is, however, an analogous n-dimensional example (for any $n \geqslant 2$) in which the two topologies are distinct and all points are polar.

The underlying space Ω in Cornea's example is the union of 3 copies $\Omega_1, \Omega_2, \Omega_3$ of the interval $[0, 1)$ such that any two of these copies are considered to be disjoint, except that the end-points 0 in all three copies are identified so as to form a single point of Ω, denoted by 0^*. The topology on Ω is defined so as to agree with the standard topology on each interval Ω_j; hence Ω is locally compact.

A function f defined in an open subset ω of Ω is called *harmonic* if, for each $j = 1, 2, 3$, the restriction f_j of f to each interval forming a component of $\omega \cap \Omega_j$ is affine, and if, moreover, $f_1'(0) + f_2'(0) + f_3'(0) = 0$ in case $0^* \in \omega$. With this sheaf of harmonic functions, Ω becomes a Brelot harmonic space with a Green function. A function f on Ω is *subharmonic* if and only if, for each $j = 1, 2, 3$, the restriction f_j of f to Ω_j is continuous and convex, and if moreover $f_1'(0) + f_2'(0) + f_3'(0) \geqslant 0$.

Let ε denote the Dirac measure on Ω at the point 0^*. The Green function $G\varepsilon = G(\cdot, 0^*)$ (suitably normalized) has the following restriction $(G\varepsilon)_j$ to Ω_j:

$$(G\varepsilon)_j(t) = \tfrac{1}{3}(1 - t), \quad 0 \leqslant t < 1.$$

Now consider the following three superharmonic functions w_j on Ω, $j = 1, 2, 3$:

$$w_j = \begin{cases} -t & \text{on } \Omega_j \\ 0 & \text{on } \Omega \setminus \Omega_j. \end{cases}$$

Simple calculations lead to the following Riesz charges:

$$\mu[w_j] = -\varepsilon,$$
$$\mu[w_j \wedge w_k] = -2\varepsilon, \quad j \neq k,$$
$$\mu[w_1 \wedge w_2 \wedge w_3] = -3\varepsilon.$$

The conclusion in Theorem 1 would thus read $-3\varepsilon \geqslant -2\varepsilon$, which is false.

Theorem 3 breaks down when we take $D_i = \Omega_i \setminus \{0^*\}$. In fact, for $z \in D_i$, $\varepsilon_z^{\Omega\setminus D_i}$ is supported by the boundary $\{0^*\}$ of D_i, and $\varepsilon_z^{\Omega\setminus D_i}(\{0^*\}) > 0$ because $\{0^*\}$ is non-polar. Consequently, any three supports S_1, S_2, S_3 as in Theorem 3 would have 0^* in common.

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