Rotations in 3-space II.

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1. Hermitian matrices in $\mathbf{C}^2$. The general form of an Hermitian matrix is

$$\begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}, \quad x_0, x_1, x_2, x_3 \in \mathbf{R}.$$

So these matrices make a *real* vector space of dimension 4. The natural basis is $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$, where $\sigma_0 = I$,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These $\sigma_1, \sigma_2, \sigma_3$ are called the Pauli matrices. Thus we can represent every Hermitian matrix as

$$H = x_0\sigma_0 + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3.$$

Notice that

$$\det H = x_0^2 - x_1^2 - x_2^2 - x_3^2,$$

the quadratic form that defines the Minkowski metric on $\mathbf{R}^4$, and

$$\operatorname{tr} H = 2x_0.$$

Exercise 1. Check the following multiplication rules for the Pauli matrices:

$$\sigma_0^2 = \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = I,$$

$$\begin{aligned} \sigma_1\sigma_2 &= i\sigma_3 = -\sigma_2\sigma_1 \\ \sigma_2\sigma_3 &= i\sigma_1 = -\sigma_3\sigma_2 \\ \sigma_3\sigma_1 &= i\sigma_2 = -\sigma_1\sigma_3 \end{aligned}$$

Let us denote

$$\mathbf{1} = \sigma_0, \quad \mathbf{i} = -i\sigma_1, \quad \mathbf{j} = -i\sigma_2, \quad \mathbf{k} = -i\sigma_3.$$

Then the rules of multiplication become

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1}, \quad \mathbf{ijk} = -\mathbf{1}.$$

These rules are carved in stone on the Broom bridge in Dublin, Ireland.

2. Hermitian matrices with trace zero. Stereogrphic projection. These matrices have the form

$$f = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}.$$

so they can be identified with vectors in $\mathbf{x} = (x_1, x_2, x_3) \in \mathbf{R}^3$ and this correspondence is one-to-one. Note that $-\det f = x_1^2 + x_2^2 + x_3^2 = \|\mathbf{x}\|$ is the standard norm in $\mathbf{R}^3$. As the trace is zero, eigenvalues of f are $\pm\sqrt{|\det f|}$. We denote them $\pm|f|$, where $|f| = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

To each eigenvalue corresponds a one-dimensional eigenspace in $\mathbf{C}^2$, and the two subspaces corresponding to the two eigenvalues $\pm|f|$ are orthogonal.

Let us assign to each non-zero Hermitian operator f with zero trace its eigenspace with positive eigenvalue. In this way we obtain a one-to-one correspondence between *rays* in $\mathbf{R}^3$ and *one-dimensional subspaces* in $\mathbf{C}^2$. (Subspace corresponding to $-f$ is the orthogonal complement of the subspace corresponding to f).

To make this correspondence explicit, let us find eigenvectors (z_1, z_2) corresponding to the positive eigenvalue. Multiplying f by a constant does not change the eigenvectors, so we may assume that that $|f| = 1$. Then we have the equations for eigenvectors corresponding to the eigenvalue 1:

$$\begin{array}{rcrcl} (x_3 - 1)z_1 & + & (x_1 - ix_2)z_2 & = & 0 \\ (x_1 + ix_2)z_1 & - & (x_3 + 1)z_2 & = & 0. \end{array}$$

We can take for example

$$z_1 = 1 + x_3, \quad z_2 = x_1 + ix_2. \tag{1}$$

Thus to each ray in $\mathbf{R}^3$ with directing vector x, $|x| = 1$ we put into correspondence a line in $\mathbf{C}^2$ which contains the vector $(z_1, z_2) \in \mathbf{C}^2$ given in (1).

The inverse map can be written as

$$x_1 = \frac{2\Re z}{1+|z|^2}, \quad x_2 = \frac{2\Im z}{1+|z|^2}, \quad x_3 = \frac{1-|z|^2}{1+|z|^2},$$

where we set $z = z_2/z_1$. Verify this! In particular, if we normalize

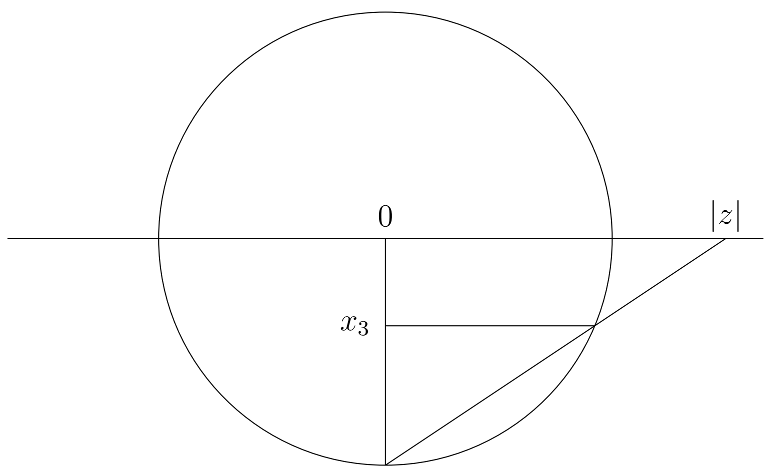
$$|z_1|^2 + |z_2|^2 = 1, \tag{2}$$

then

$$x_1 = z_1^* z_2 + z_1 z_2^*, \quad x_2 = \frac{1}{i}(z_1^* z_2^* - z_1 z_2^*), \quad x_3 = |z_1|^2 - |z_2|^2. \tag{3}$$

These formulas give $|x| = 1$.

The map from the unit sphere in $\mathbf{R}^3$ to the complex z-plane given by equations (1) is called the *stereographic projection*; it can be visualized as seen in the figure.



What happens to the angles under this correspondence? The angle between two unit vectors in $\mathbf{R}^3$ is defined by

$$\cos\theta = (x, y), \quad \theta \in [0, \pi]. \tag{4}$$

Suppose that $\mathbf{z} = (z_1, z_2)$ is a unit vector obtained by normalization of the vector in (1), and $\mathbf{w} = (w_1, w_2)$ a unit vector corresponding to to $y \in \mathbf{R}^3$ via (1). Then the angle α between the lines through $\mathbf{z}$ and $\mathbf{w}$ satisfies

$$\cos\alpha = |z_1^* w_1 + z_2^* w_2|. \tag{5}$$

We claim that

$$\alpha = \theta/2, \quad \text{that is} \quad \cos\theta = 2\cos^2\alpha - 1. \tag{6}$$

To check the claim, use (5), (4) and (3) and the similar formula for (y_1, y_2, y_3) in terms of (w_1, w_2). We have

$$\begin{array}{rcl} \cos\theta & = & x_1y_1 + x_2y_2 + x_3y_3 \\ & = & (z_1^*z_2 + z_1z_2^*)(w_1^*w_2 + w_1w_2^*) - (z_1^*z_2 - z_1z_2^*)(w_1^*w_2 - w_1w_2^*) \\ & + & (|z_1|^2 - |z_2|^2)(|w_1|^2 - |w_2|^2) \\ & = & 4\Re(z_1^*z_2w_1w_2^*) + |z_2|^2|w_2|^2 + |z_1|^2|w_1|^2 - |z_1|^2|w_2|^2 - |z_2|^2|w_1|^2. \end{array}$$

and

$$\begin{array}{rcl} 2\cos^2\alpha - 1 & = & 2|z_1^*w_1 + z_2^*w_2|^2 - 1 \\ & = & 2|z_1|^2|w_1|^2 + 2|z_2|^2|w_2|^2 + 4\Re(z_1^*z_2w_1w_2^*) - 1. \end{array}$$

After a simplification we see using (2) that these two expressions are equal. This proves (6).

3. Unitary matrices with determinant 1 **in $\mathbf{C}^2$.** Let

$$Q = \left(\begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right).$$

Q is unitary if $Q^{-1} = Q^*$. Together with $\det Q = 1$ this gives

$$\left(\begin{array}{cc} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{array} \right) = \left(\begin{array}{cc} \delta & -\beta \\ -\gamma & \alpha \end{array} \right),$$

so $\delta = \alpha^*,\ \gamma = -\beta^*$, and the general form of a unitary matrix with determinant 1 is

$$Q = \left(\begin{array}{cc} \alpha & \beta \\ -\beta^* & \alpha^* \end{array} \right) \quad \text{where} \quad \alpha\alpha^* + \beta\beta^* = 1.$$

The set of such matrices is called $SU(2)$, which means "special unitary 2 by 2".

4. Action of $SU(2)$ on $\mathbf{R}^3$. Let us represent points $\mathbf{x} = (x_1, x_2, x_3)$ in $\mathbf{R}^3$ by traceless Hermitian matrices

$$P = x_1\sigma_2 + x_2\sigma_2 + x_3\sigma_3.$$

Let us fix a unitary matrix $Q \in SU(2)$ and consider the linear operator

$$P \mapsto L_Q(P) = QPQ^* = QPQ^{-1},$$

the last equality holds because Q is unitary.

Then $\det L(P) = \det P$, and $\operatorname{tr} L(P) = \operatorname{tr} P$, because both the determinant and the trace do not change under a similarity transformation. So L_Q is indeed well defined: it maps the space of traceless Hermitian matrices to itself. Moreover, as we've seen in section 1,

$$\det P = -x_1^2 - x_2^2 - x_3^2,$$

which is $-\|x\|$, the Euclidean norm. Thus our operator L_Q is orthogonal (with respect to the standard dot product in $\mathbf{R}^3$).

Thus to each unitary matrix Q with determinant 1 corresponds an orthogonal operator of $\mathbf{R}^3$. Let us find the explicit 3×3 matrix of this orthogonal operator.