Rotations in 3-space

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> These facts of the combinations of rotations, and what they produce, are hard to grasp intuitively. It is rather strange, because we live in three dimensions, but it is hard for us to appreciate what happens if we turn this way and then that way. Perhaps, if we were fish or birds and had a real appreciation of what happens when we turn somersaults in space, we could more easily appreciate these things.
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> Feynman's lectures on physics, vol. 3, 6.5.

> $O(3)$ pervades all the essential properties of the physical world. But we remain intellectually blind to this symmetry, even if we encounter it frequently and use it in everyday life, for instance when we experience or engender mechanical movements, such as walking. This is due in part to the non commutativity of $O(3)$, which is difficult to grasp.
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> M. Gromov, Notices AMS, 45, 846-847.

Rotations in 3-space are more complicated than rotations in a plane. One reason for this is that rotations in 3-space in general do not commute. (Give an example!)

We consider rotations with fixed center which we place at the origin. Then rotations are linear transformations of the space represented by orthogonal matrices with determinant 1.

Orthogonality means that all distances and angles are preserved. The determinant of an orthogonal matrix is always 1 or -1. (Think why.) The condition that determinant equals 1 means that *orientation* is preserved. An orthogonal transformation with determinant -1 will map a right shoe onto a left shoe; such orthogonal transformations are called sometimes *improper rotations*, we cannot actually perform them on real objects. (No matter how you rotate a left shoe it will never become a right shoe).

We begin with simple properties of rotations in 3-space.

Theorem 1. *Let $A \neq I$ be a 3×3 rotation matrix. Then* 1 *is an eigenvalue of* A. *Two other eigenvalues are either both equal to* -1 *or they are distinct complex conjugate numbers of absolute value* 1.

The first statement means that there is a vector $x \neq 0$ which remains fixed, $Ax = x$. So each vector of the one-dimensional subspace spanned by this x remains fixed, that is every rotation has an *axis*. This is not so in dimension 2. A (non-identiry) rotation in dimension 2 displaces every non-zero vector. Give an example of rotation in dimension 4 which displaces every non-zero vector. It is of crucial importance that our space has an odd dimension.

Exercise: give an example of rotation in dimension 4 which moves every non-zero vector.

Proof of Theorem 1. First notice that all eigenvalues of an orthogonal matrix have absolute value 1. Indeed,

$$Ax = \lambda x, \quad \|x\|^2 = (x, x) = (Ax, Ax) = |\lambda|^2 (x, x) = |\lambda|^2 \|x\|^2,$$

and $x \neq 0$, so $|\lambda| = 1$.

As the matrix is real, its non-real eigenvalues come in conjugate pairs.

Now the characteristic equation $p(\lambda) = \det(A - \lambda I) = 0$ has degree 3, and $p(\lambda) = -\lambda^3 + \ldots$. As $p(0) = \det A = 1$, the graph of $p(\lambda)$ has to cross the λ-axis at some positive value, so we have at least one positive eigenvalue λ_1. As $|\lambda_1| = 1$ we conclude that $\lambda_1 = 1$. $\square$

The plane P perpendicular to the axis of rotation is invariant (because the rotation preserves all angles). It follows that the restriction of our rotation to this plane P is a rotation of P. So a rotation is completely characterized by two pieces of data: the direction of the axis of rotation and the angle of rotation about this axis.

It is convenient to combine these data into one vector whose direction specifies the axis and the whose length equals to the angle of rotation. So the angle of rotation is always between 0 and π, and the direction of the vector along the axis is chosen in such a way that it makes the "right screw" with the rotation. We denote by $g(a)$ the rotation that corresponds to the vector a, $\|a\| \leq \pi$.

Notice that the correspondence $a \mapsto g(a)$ is not one-to-one: if $\|a\| = \pi$ then $g(a) = g(-a)$.

Given the axis and the angle of a rotation, how to write the matrix of this rotation?

To answer this question, let us consider all skew symmetric 3×3 matrices, which we will write in the form

$$A(a) = a_1 A_1 + a_2 A_2 + a_3 A_3,$$

where $a = (a_1, a_2, a_3)$,

$$A(a) = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} = a_1 A_1 + a_2 A_2 + a_3 A_3,$$

and

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so that A_1, A_2 and A_3 form a basis in the space of all 3×3 skew-symmetric matrices, and a_1, a_2, a_3 are the coordinates of a skew symmetric matrix $A(a)$ with respect to this basis.

Exercise: verify that A_1, A_2 and A_3 satisfy the relations

$$\begin{aligned} A_1 A_2 - A_2 A_1 &= A_3 \\ A_2 A_3 - A_3 A_2 &= A_1 \\ A_3 A_1 - A_1 A_3 &= A_2. \end{aligned} \quad .$$

The expressions in the left hand side are called *commutators*, in general, the commutator of two matrices A and B is defined by $[A, B] = AB - BA$.

Exercise: verify that for every two vectors a and b, we have $[A(a), A(b)] = A(a \times b)$, where $a \times b$ is the *cross product*.

So the rules of multiplication of matrices $A(a)$ are the same as the rules of cross multiplication of vectors a.

Let us compute the exponent $\exp(A_3 t)$. One can do this in the usual way, by finding the eigenvalues and eigenvectors, but in this particular case it is easier to apply the definition directly. Indeed, we have (verify!)

$$A_3^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -I', \quad A_3^3 = -A_3, \quad A_3^4 = I',$$

and so on. So, using the definition of the exponential, and grouping the even and odd terms, we obtain

$$e^{\alpha A_3} = \sum_{n=0}^{\infty} \frac{\alpha^n A_3^n}{n!} = I + I' \sum_{k=1}^{\infty} \frac{(-1)^k \alpha^{2k}}{(2k)!} + A_3 \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \alpha^{2k-1}}{(2k-1)!} =$$

$$= I + I'(\cos\alpha - 1) + A_3 \sin\alpha,$$

that is

$$e^{\alpha A_3} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1}$$

Thus $\exp(\alpha A_3)$ is a rotation about z-axis by the angle α. Similarly you can verify that $\exp(\alpha A_1)$ and $\exp(\alpha A_2)$ are rotations by the angle α about the x and y axes, respectively. The axes are supposed to be oriented in their positive directions. Check that the right screw rule is observed!

Now we can establish our main result:

Theorem 2. *Rotation corresponding to the vector $a = (a_1, a_2, a_3)$ is given by the formula*

$$g(a) = \exp(a_1 A_1 + a_2 A_2 + a_3 A_3). \tag{2}$$

Proof. Let $g(a)$ be the rotation corresponding to the vector $a = (a_1, a_2, a_3)$. When two of the coordinates of a equal zero, we have an explicit formula for $g(a)$, for example, if $a_1 = a_2 = 0$, this is a rotation about z axis, so it is given by (1) with $\alpha = a_3$. The formulas for the other two axes are similar. Differentiating these formulas we obtain

$$\frac{\partial g(a)}{\partial a_1}|_{a=0} = \frac{\partial g(a_1, 0, 0)}{\partial a_1}|_{a_1=0} = \frac{\partial}{\partial a_1} e^{a_1 A_1}|_{a_1=0} = A_1, \tag{3}$$

and similarly

$$\frac{\partial g(a)}{\partial a_2}|_{a=0} = A_2, \quad \frac{\partial g(a)}{\partial a_3}|_{a=0} = A_3. \tag{4}$$

Now, let $n = (n_1, n_2, n_3)$ be a unit vector in some direction, then $g(n\alpha)$ and $g(n\beta)$ are rotations about the same axis by angles α and β respectively, so evidently we have

$$g(n(\alpha + \beta)) = g(n\alpha)g(n\beta).$$

Differentiating this identity with respect to β, then setting $\beta = 0$ and using the above relations (3) and (4), we obtain

$$\frac{d}{d\alpha} g(n\alpha) = (n_1 A_1 + n_2 A_2 + n_3 A_3) g(n\alpha).$$

On the other hand, if we denote the right hand side of (2) by $f(a_1, a_2, a_3)$, and put $a = (a_1, a_2, a_3) = n\alpha$, then the differentiation with respect to α gives

$$\frac{d}{d\alpha} f(n\alpha) = (n_1 A_1 + n_2 A_2 + n_3 A_3) f(n\alpha).$$

Thus $f(n\alpha)$ and $g(n\alpha)$ satisfy the same differential equation and the same initial condition (namely, $g(n0) = f(n0) = I$), and we conclude from the uniqueness theorem for differential equations that $g = f$. This completes the proof.