Rotations in 3-space

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These facts of the combinations of rotations, and what they
produce, are hard to grasp intuitively. It is rather strange,
because we live in three dimensions, but it is hard for us to
appreciate what happens if we turn this way and then that way.
Perhaps, if we were fish or birds and had a real appreciation of
what happens when we turn somersaults in space, we could more
easily appreciate these things.

Feynman's lectures on physics, vol. 3, 6.5.

$O(3)$ pervades all the essential properties of the physical world.
But we remain intellectually blind to this symmetry, even if we
encounter it frequently and use it in everyday life, for instance
when we experience or engender mechanical movements, such as
walking. This is due in part to the non commutativity of $O(3)$,
which is difficult to grasp.

M. Gromov, Notices AMS, 45, 846-847.

Rotations in 3-space are more complicated than rotations in a plane. One
reason for this is that rotations in 3-space in general do not commute. (Give
an example!)

We consider rotations with fixed center which we place at the origin. Then
rotations are linear transformations of the space represented by orthogonal
matrices with determinant 1.

Orthogonality means that all distances and angles are preserved. The
determinant of an orthogonal matrix is always 1 or -1. (Think why.) The

condition that determinant equals 1 means that *orientation* is preserved. An orthogonal transformation with determinant -1 will map a right shoe onto a left shoe; such orthogonal transformations are called sometimes *improper rotations*, we cannot actually perform them on real objects. (No matter how you rotate a left shoe it will never become a right shoe).

We begin with simple properties of rotations in 3-space.

Theorem 1. *Let $A \neq I$ be a 3×3 rotation matrix. Then 1 is an eigenvalue of A. Two other eigenvalues are either both equal to -1 or they are distinct complex conjugate numbers of absolute value 1.*

The first statement means that there is a vector $x \neq 0$ which remains fixed, $Ax = x$. So each vector of the one-dimensional subspace spanned by this x remains fixed, that is every rotation has an *axis*. This is not so in dimension 2. A (non-identity) rotation in dimension 2 displaces every non-zero vector. It is of crucial importance that our space has an odd dimension.

Exercise: give an example of rotation in dimension 4 which moves every non-zero vector.

Proof of Theorem 1. First notice that all eigenvalues of an orthogonal matrix have absolute value 1. Indeed,

$$Ax = \lambda x, \quad \|x\|^2 = (x, x) = (Ax, Ax) = |\lambda|^2(x, x) = |\lambda|^2 \|x\|^2,$$

and $x \neq 0$, so $|\lambda| = 1$.

As the matrix is real, its non-real eigenvalues come in conjugate pairs.

Now the characteristic equation $p(\lambda) = \det(A - \lambda I) = 0$ has degree 3, and $p(\lambda) = -\lambda^3 + \ldots$. As $p(0) = \det A = 1$, the graph of $p(\lambda)$ has to cross the λ-axis at some positive value, so we have at least one positive eigenvalue λ_1. As $|\lambda_1| = 1$ we conclude that $\lambda_1 = 1$. □

The plane P perpendicular to the axis of rotation is invariant (because the rotation preserves all angles). It follows that the restriction of our rotation to this plane P is a rotation of P. So a rotation is completely characterized by two pieces of data: the direction of the axis of rotation and the angle of rotation about this axis.

It is convenient to combine these data into one vector whose direction specifies the axis and the whose length equals to the angle of rotation. So the angle of rotation is always between 0 and π, and the direction of the

vector along the axis is chosen in such a way that it makes the "right screw" with the rotation. We denote by $g(a)$ the rotation that corresponds to the vector a, $\|a\| \leq \pi$.

Notice that the correspondence $a \mapsto g(a)$ is not one-to-one: if $\|a\| = \pi$ then $g(a) = g(-a)$.

Given the axis and the angle of a rotation, how to write the matrix of this rotation?

To answer this question, let us consider all skew symmetric 3×3 matrices, which we will write in the form

$$A(a) = a_1 A_1 + a_2 A_2 + a_3 A_3,$$

where $a = (a_1, a_2, a_3)$,

$$A(a) = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} = a_1 A_1 + a_2 A_2 + a_3 A_3,$$

and

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so that A_1, A_2 and A_3 form a basis in the space of all 3×3 skew-symmetric matrices, and a_1, a_2, a_3 are the coordinates of a skew symmetric matrix $A(a)$ with respect to this basis.

Exercise: verify that A_1, A_2 and A_3 satisfy the relations

$$A_1 A_2 - A_2 A_1 = A_3$$
$$A_2 A_3 - A_3 A_2 = A_1 \quad .$$
$$A_3 A_1 - A_1 A_3 = A_2.$$

The expressions in the left hand side are called *commutators*, in general, the commutator of two matrices A and B is defined by $[A, B] = AB - BA$.

Exercise: verify that for every two vectors a and b, we have $[A(a), A(b)] = A(a \times b)$, where $a \times b$ is the *cross product*.

So the rules of multiplication of matrices $A(a)$ are the same as the rules of cross multiplication of vectors a.

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Let us compute the exponent $\exp(A_3 t)$. One can do this in the usual way, by finding the eigenvalues and eigenvectors, but in this particular case it is easier to apply the definition directly. Indeed, we have (verify!)

$$A_3^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -I', \quad A_3^3 = -A_3, \quad A_3^4 = I',$$

and so on. So, using the definition of the exponential, and grouping the even and odd terms, we obtain

$$e^{\alpha A_3} = \sum_{n=0}^{\infty} \frac{\alpha^n A_3^n}{n!} = I + I' \sum_{k=1}^{\infty} \frac{(-1)^k \alpha^{2k}}{(2k)!} + A_3 \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \alpha^{2k-1}}{(2k-1)!} =$$

$$= I + I'(\cos\alpha - 1) + A_3 \sin\alpha,$$

that is

$$e^{\alpha A_3} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1}$$

Thus $\exp(\alpha A_3)$ is a rotation about z-axis by the angle α. Similarly you can verify that $\exp(\alpha A_1)$ and $\exp(\alpha A_2)$ are rotations by the angle α about the x and y axes, respectively. The axes are supposed to be oriented in their positive directions. Check that the right screw rule is observed!

Now we can establish our main result:

Theorem 2. *Rotation corresponding to the vector* $a = (a_1, a_2, a_3)$ *is given by the formula*

$$g(a) = \exp(a_1 A_1 + a_2 A_2 + a_3 A_3). \tag{2}$$

Proof. Let us find the eigenvalues of the skew symmetric matrix

$$B := a_1 A_1 + a_2 A_2 + a_3 A_3 = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}.$$

We obtain the characteristic polynomial

$$\det(B - \lambda I) = -\lambda^2 - (a_1^2 + a_2^2 + a_3^2)\lambda,$$

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so the eigenvalues of B are $0, \pm i\|a\|$, and by the spectral mapping theorem the eigenvalues of $g(a) = \exp B$ are

$$1, \quad \exp(\pm i\|a\|).$$

Now it is easily verified that $(a_1, a_2, a_3)^T$ is an eigenvector of B corresponding to eigenvalue 0. So it is the eigenvector of $\exp B$ with eigenvalue 1. Then $\exp B$ is orthogonal (as the exponent of any skew-symmetric matrix), has determinant 1 and eigenvelues 1, $e^{\pm i\|a\|}$, so it is the rotation with axis a by angle $\|a\|$.