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Distribution of zeros of some real polynomials and iteration theory

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Theorem. Let f be a real polynomial of degree m. Then $f'+f^2$ has at most $m+1$ real zeros.

Under the additional condition that f has only real zeros this is a reformulation of problem 182 from [PS , part IV].

This Theorem is a special case of famous conjecture of A. Wiman concerning the distribution of zeros of second derivative of an entire function with real zeros. An account of the subject and references are given in [H].

Recently T. Sheil-Small [S] proved Wiman's conjecture including the Theorem stated above. His main tool was the auxiliary function $P(z)=z-1/f(z)$. Its critical points coincide with the zeros of $f'+f^2$. Using the same auxiliary function we give a very short proof of the Theorem based on iteration theory of rational functions [F].

Proof. P is a real rational function and

$$P(z)=z+cz^{-m}+O(z^{-m-1}),\ z\to\infty,\ c\neq 0. \quad (1)$$

The idea is to iterate P. The infinity is a degenerate neutral fixed point of order $m+1$. It follows from Fatou theory that ∞ attracts at least $m+1$ critical points $c_1,\dots,c_{m+1}$ of P. The trajectories of these points have asymptotic directions $l_1,\dots,l_{m+1}$ which are equally spaced (i. e. the angle between l_j and l_{j+1} is equal to $2\pi/(m+1)$).

Only two of these directions may be parallel to the real axis. Thus at least $m-1$ of the points $c_1, \dots, c_{m+1}$ are non-real, because the real axis is P -invariant. This proves the Theorem.

Remark. Consider the entire function $F(z)=\exp\int_0^z f(\zeta)d\zeta$. Iterating $P(z)=z-F(z)/F'(z)$ is the Newton's method of searching roots of F. But F has no roots at all!

References

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