Schwarzian derivatives of rational functions

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1. Properties of the Schwarzian derivative,
\[ \{y, z\} = \frac{y'''}{y'} - \frac{3}{2} \left( \frac{y''}{y'} \right)^2, \quad \text{where} \quad ' = \frac{d}{dz}. \]

1.1 Consider the second order linear differential equation
\[ w'' + Gw = 0. \quad (1) \]
If \( w \) and \( w_1 \) are two linearly independent solutions of (1), then \( y = \frac{w_1}{w} \) satisfies
\[ \{y, z\} = 2G. \quad (2) \]
To verify this, we recall that the Wronskian \( w_1'w - w_1w' = c \) is constant. So we have
\[ y = \frac{w_1}{w}, \quad y' = \frac{c}{w^2}, \quad y'' = -2c \frac{w'}{w^3}, \]
and
\[ y''' = -2c \frac{w''w - 3w'^2}{w^4}, \]
so
\[ \{y, z\} = \frac{y'''}{y'} - \frac{3}{2} \left( \frac{y''}{y'} \right)^2 = -2 \frac{w''}{w} = 2G. \]
In the opposite direction, if \( y \) satisfies (2) then
\[ w = \frac{1}{\sqrt{y}}, \quad \text{and} \quad w_1 = \frac{y}{\sqrt{y'}}. \]

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satisfy (1), which is verified by direct computation.

1.2 As a corollary we obtain

\[ \{y_1, z\} = \{y_2, z\} \text{ if and only if } y_1 = L \circ y_2, \]

where \( L \) is a fractional-linear transformation.

1.3 If \( y \) is a meromorphic function of \( z \) then \( \{y, z\} \) is meromorphic, moreover, it is holomorphic except at the critical points of \( y \), where it has poles of exactly second order. This can be verified directly, or using 1.1.

2. **Regular singular points of the equation (1).** A point \( z_0 \) is called singular if \( G \) is not holomorphic at \( z_0 \). A singular point is regular (see [5, 6]) if \( G \) has a pole of order at most 2 at this point. Making the change of variable \( w(z) = v(1/z), \ G(z) = H(1/z), \) and \( \zeta = 1/z, \) we obtain

\[ \zeta^4 v'' + 2\zeta^3 v' + H(\zeta) v = 0, \]

and the singular point at \( \infty \) is regular iff

\[ G(z) = O(z^{-2}), \quad z \to \infty. \]

If one solution of (2) is meromorphic in some domain \( D \) then all solutions are meromorphic in \( D \) (because all solutions are obtained from one of them by a fractional-linear transformation), and all singularities of (1) in \( D \) are regular in this case.

2.1 Suppose that 0 is a regular singular point of the equation (1). To use the general theory of regular singular points (see, for example, [5, 6]) we write the equation in the form

\[ z^2 w'' + P(z) w = 0, \quad \text{where} \quad P(z) = a_0 + a_1 z + a_2 z^2 + \ldots. \]

Put \( F(r) = r(r - 1) + a_0, \) this is called the characteristic polynomial of (5), corresponding to the point 0. Let \( r_1 \) and \( r_2 \) be the two solutions of the indicial equation

\[ F(r) = r(r - 1) + a_0 = (r - r_1)(r - r_2) = 0. \]

The following cases may occur:
a) If \( r_1 - r_2 \) is not an integer, equation (5) has two linearly independent convergent power series solutions of the form \( w_j(z) = z^{r_j}Q_j(z), \ j = 1, 2. \) Here \( Q_j \) are McLauren series (containing only non-negative integral powers of \( z \)).

b) If \( r_1 - r_2 \) is an integer, then \( r_1 \) and \( r_2 \) are real, because \( r_1 + r_2 = 1 \) by Vieta's theorem. We label them so that \( r_1 \geq r_2. \) Then, if \( r_1 - r_2 \neq 0, \) there are two linearly independent solutions of the form

\[
w_1(z) = z^{r_1}Q_1(z) \quad \text{and} \quad w_2(z) = z^{r_2}Q_2(z) + Cw_1(z) \log z,
\]

where \( C \) is a constant, \( Q_1 \) and \( Q_2 \) are McLauren series. If \( r_1 = r_2 \) there are two linearly independent solutions of the form

\[
w_1(z) = z^{r_1}Q_1(z) \quad \text{and} \quad w_2(z) = w_1(z) \log z + z^{r_1}Q_2(z),
\]

so a logarithm is always present in the general solution in this case.

2.2 We are interested in the case, when

\[
r_1 = r_2 + 2, \quad \text{and} \quad C = 0. \tag{7}
\]

(The conditions that \( r_1 - r_2 \) is a positive integer, and \( C = 0 \) are necessary and sufficient for the ratio of two linearly independent solutions to be meromorphic at 0. The additional condition \( r_1 - r_2 = 2 \) ensures that this meromorphic ratio has a simple critical point at 0.)

The first of these conditions (7), together with \( r_1 + r_2 = 1, \) imply that

\[
r_2 = -1/2, \quad r_1 = 3/2 \quad \text{and thus by} \ (6) \tag{8}
\]

\[
a_0 = P(0) = -3/4. \tag{9}
\]

To find the necessary and sufficient condition for \( C = 0 \) in (7) in terms of coefficients \( a_j \) of \( P \) in (5), we plug the power series \( w(z) = z^r(c_0 + c_1z + \ldots) \) with \( r = r_2 = -1/2 \) and \( c_0 = 1 \) into (5), where \( a_0 = -3/4, \) according to (8). We obtain

\[
r(r - 1) + a_0 = 0, \tag{10}
\]

\[
[(r + 1)r + a_0]c_1 = -a_1c_0, \tag{11}
\]

\[
[(r + 2)(r + 1) + a_0]c_2 = -a_2c_0 - a_1c_1, \tag{12}
\]

\[
\ldots,
\]

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and in general
\[ F(r + n)c_n = \text{polynomial in } c_0, \ldots, c_{n-1}, \quad n = 0, 1, 2, \ldots, \]  
where \( F \) is defined in (6). Equation (10) is satisfied because \( r = -1/2 \), and \( a_0 = -3/4 \). Equation (11) (with \( r = -1/2 \), \( a_0 = -3/4 \) and \( c_0 = 1 \)) then implies \( c_1 = a_1 \), and equation (12), whose left hand side is 0, implies
\[ a_1^2 + a_2 = 0. \]  
Thus if (5) has a power series solutions with properties (7), then (9) and (14) are satisfied. The converse is also true: if these two conditions are satisfied, than all coefficients \( c_j \) can be successively found from (13), because \( F(r + n) \neq 0 \) for \( n \geq 3 \). Thus we have

**Proposition.** In order that the ratio of two linearly independent solutions of (5) be meromorphic at 0, and have a simple critical point there, it is necessary and sufficient that conditions (9) and (14) be satisfied.

3. **Schwarzian derivatives of rational functions.** We say that a finite critical point \( z_0 \) of a rational function \( f \) is simple if \( f''(z_0) \neq 0 \). (If \( f(z_0) = \infty \), this has to be modified to \( 1/f''(z_0) \neq 0 \).)

**Theorem 1.** Suppose that \( f \) is a rational function whose finite critical points \( z_1, \ldots z_n \) are simple. Then
\[ \frac{1}{2} \{f, z\} = -\frac{3}{4} \sum_{k=1}^{n} \frac{1}{(z - z_k)^2} + \sum_{k=1}^{n} \frac{x_k}{z - z_k}, \]  
where
\[ x_m^2 + \sum_{k \neq m} \frac{x_k}{z_m - z_k} = \frac{3}{4} \sum_{k \neq m} \frac{1}{(z_m - z_k)^2}, \quad m = 1, \ldots, n. \]  

**Remark.** Substitution
\[ x_m = y_m - \frac{1}{2} \sum_{k \neq m} \frac{1}{z_m - z_k} \]  
simplifies equations (16) to
\[ y_m^2 = \sum_{k \neq m} \frac{y_k - y_m}{z_k - z_m}. \]
This equivalent form of equations (16) was obtained in [3].

**Proof.** Let \( f \) be a rational function with simple finite critical points \( z_1, \ldots, z_n \). Put \( G = (1/2) \{ f, z \} \). Then \( G \) is a rational function with only poles at \( z_k \), all these poles are double by 1.4, and \( G \) satisfies (4). In particular, \( G \) has the form
\[
G(z) = \sum_{k=1}^{n} \frac{b_k}{(z - z_k)^2} + \frac{x_k}{z - z_k}.
\]

Now the ratio of two linearly independent solutions of (1) with this \( G \) has to be a rational function. An inspection of the cases a) and b) in section 2.1 shows that this will be the case only if at each singular point \( z_k \) we have \( r_1 - r_2 \) an integer and \( C = 0 \) (so that there are no logarithms). From the additional condition that each \( z_m \) is a simple critical point of \( f \) we deduce that \( r_1 - r_2 = 2 \). Thus we have (9) with \( a_0 = b_m \) and (14) with \( a_0 = -3/4 \), \( a_1 = x_m \) for each singular point \( z_m \). This gives \( b_m = -3/4 \) and (16).

**Theorem 2.** Let \( z_1, \ldots, z_n \) be distinct complex numbers, \((x_1, \ldots, x_n)\) a solution of (16) and \( G(z) \) the rational function defined by the right hand side of (15). Then:
\[
\sum_{k=1}^{n} x_k = 0, \tag{17}
\]
and
\[
\sum_{k=1}^{n} x_k z_k - \frac{3}{4} n = \frac{1 - q^2}{4}, \tag{18}
\]
where \( q \) is a positive integer.

Furthermore, the general solution of equation (11) with this \( G \) is a rational function of degree \((n + q + 1)/2\) having simple critical points at \( z_1, \ldots, z_n \) and a critical point of multiplicity \( q - 1 \) at infinity.

**Proof.** If (16) is satisfied then the differential equation (2) defines a meromorphic function \( y \) in \( C \). All critical points of \( y \) in \( C \) are simple and occur exactly at \( z_1, \ldots, z_n \). By definition of \( G \), we have
\[
G(z) \sim c/z, \quad z \to \infty.
\]
Suppose first that \( c \neq 0 \). Then, by the well-known asymptotic analysis of the equation (1) (see, for example, [5]) we conclude that \( y \) is a meromorphic function of order \( 1/2 \) and has one asymptotic value. In addition, it has
finally many critical points. It is clear that such function cannot exist. The conclusion is that \( c = 0 \), which is equivalent to (4). Computing this \( c \) from (15) we obtain (17).

**Remark.** We proved that (16) implies (17). Can one prove this fact in a more direct way?

As infinity is now a regular singular point of (1), we conclude that our meromorphic function \( y \) cannot have an essential singularity, so it is rational. This implies for the equation (3), that the difference between the exponents \( r_1 \) and \( r_2 \) is a positive integer, and there are no logarithms in the formal solutions. Let \( \lim_{z \to \infty} z^2 G(z) = a \). The indicial equation of (3) at infinity is

\[ r^2 + r + a = 0, \]

and its solutions are

\[ r_1 = \frac{-1 + \sqrt{1 - 4a}}{2} \quad \text{and} \quad r_2 = \frac{-1 - \sqrt{1 - 4a}}{2}. \]

Now \( q = r_1 - r_2 \) is a positive integer and we conclude that

\[ a = (1 - q^2)/4. \]

Computing \( a \) from (15) we obtain (18).

The critical point of \( y \) at infinity has order \( q - 1 \), so the total number of critical points on the Riemann sphere is \( n + q - 1 \), so \( y \) has degree \((n + q + 1)/2\).

4. Let us call two rational functions \( f_1 \) and \( f_2 \) equivalent if \( f_1 = L \circ f_2 \) for some fractional-linear \( L \) by 1.2. Two rational functions are equivalent if they have the same Schwarzian derivative. An equivalence class contains a real function if and only if the Schwarzian derivative of functions of this class is real. Indeed, if there is a real function in a class than its Schwarzian derivative is real. In the opposite direction, suppose that the Schwarzian derivative \( G/2 \) of a class is real. Then the differential equation

\[ \{y, z\} = G/2 \]

has at least one real solution \( y_0 \) (take any real initial conditions to solve the Cauchy problem). This means that there is a real function in the class, namely \( y_0 \).
5. Suppose that \( q = 1 \) in (18). Then \( a = 0 \), and the condition of absence of logarithms in the formal solution at infinity gives

\[
\sum_{k=1}^{n} x_k z_k^2 = \frac{3}{2} \sum_{k=1}^{n} z_k.
\]

It is clear that \( q \leq n + 1 \), as a rational function cannot have more than half of its critical points at infinity. In the extremal case, \( q = n + 1 \), \( y \) is a “polynomial” (up to a fractional-linear transformation) of degree \( n + 1 \), and such solution of (16) is unique. The number of solutions with any fixed \( q \) can be counted using the method of [2], for example, it is the Catalan number for \( q = 1 \). In general, for a fixed \( \geq 2 \) and \( n \), it is the number of chord diagrams (degenerate nets, using the terminology of [2]) with \( n + 1 \) vertices on the unit circle, each vertex except one is the endpoint of exactly one chord, and the exceptional vertex is the endpoint of \( q - 2 \) chords. The sum of all these numbers, for \( 1 \leq q \leq n + 1 \) gives the total number of solutions of (16). If all \( z_k \) are real, all these solutions are real by the result of [1].

References


