Problem: prove that the series

$$\sum_{n=1}^{\infty}(-1)^{n+1}n^{-s} \quad \text{is convergent.}$$

Proof. We use a version of the trick called Abel's transform (a.k.a., summation by parts). Suppose that the series $\sum b_n$ is convergent, and consider its "tails"

$$B_n = \sum_{k=n}^{\infty} b_k \to 0 \quad \text{as} \quad n \to \infty., \tag{1}$$

Then for arbitrary sequence a_n, we have

$$S_{k,m} := \sum_{n=k}^{m} a_n b_n = \sum_{n=k}^{m} a_n(B_n - B_{n+1}) = a_k B_k - a_m B_{m+1} + \sum_{n=k}^{m-1}(a_{n+1} - a_n)B_{n+1}. \tag{2}$$

Now we write $s = \sigma + it$,

$$(-1)^{n+1}n^{-s} = (-1)^{n+1}n^{-\sigma/2}n^{-\sigma/2-it} =: b_n a_n,$$

where $b_n = (-1)^{n+1}n^{-\sigma/2}$, and $a_n = n^{-\sigma/2-it} =: n^{-z}$, where $z = \sigma/2 + it$.

Then the series $\sum b_n$ is convergent (as an alternating series, whose term tends to zero). To apply the summation formula (2), we estimate

$$|a_{n+1} - a_n| \;=\; |(n+1)^{-z} - n^{-z}| = \left| z \int_n^{n+1} x^{-z-1} dx \right| \le \tag{3}$$

$$\le \; |z| \int_n^{n+1} x^{-\sigma/2-1} = \frac{2|z|}{\sigma}(n^{-\sigma/2} - (n+1)^{-\sigma/2}). \tag{4}$$

Now we estimate (2). The first two summands in the RHS clearly tend to sero, when $m, k \to \infty$, so we ignore them. For the rest we use $|B_n| < \epsilon$ (see (1)), and (3):

$$|\sum_{n=k}^{m} a_n b_n| \le \frac{2\epsilon|z|}{\sigma} \sum_{n=k}^{m-1}(n^{-\sigma/2} - (n+1)^{-\sigma/2}).$$

The last sum is "telescoping", so it is equal to $k^{-\sigma/2} - m^{-\sigma/2}$, and the whole expression tends to zero, as m and k tend to infinity.