Infinite series

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We are usually taught that Calculus is integration and differentiation. This is a very simplified picture. Calculus is about "infinite procedures", and infinite series and infinite products also always played an important role.

The earliest evidence about infinite series is the paradox which is due to the ancient Greek philosopher Zeno and which is called "Achilles and the Turtle". I recall the paradox.

Achilles[1] and the Turtle[2] begin a running competition. They start simultaneously, but in the beginning, the Turtle is 1 length unit ahead of Achilles, and for simplicity we assume that they are running at constant speeds a for Achilles and $b < a$ for the Turtle.

The question is whether Achilles will ever pass the Turtle and when. I am sure that Zeno could compute the answer: the difference in speeds is $a - b$, and at time $T = 1/(a - b)$ Achilles will overtake the Turtle.

But Zeno proposes to argue differently. Achilles will reach the starting point of the Turtle at the time $t_1 = 1/a$. In this time the Turtle will move b/a units ahead. To reach this new place, Achilles will need time $t_2 = b/a^2$. But in this time, the Turtle will reach new place b^2/a^2 units ahead of the previous place... And so on. It is clear that this argument can be continued indefinitely, and Zeno's conclusion was that Achilles will never pass the Turtle. This is called a Zeno's paradox (he had few more of the same sort).

As I understand, the real conclusion from this which Zeno made was not that Achilles will never overtake the Turtle in a real competition, but rather

[1] The legendary warrior. Presumably also a sportsman, he is supposed to be able to run very fast.

[2] This Turtle is supposed to be a slow runner.

that “motion is incomprehensible”.

Since then, a great progress was made in understanding motion, and in mathematics. We know that an infinite series can have a well-defined, finite sum. From our modern point of view, Zeno’s argument represents the time T as a sum of an *infinite series*

$$T = t_1 + t_2 + \ldots = \frac{1}{a} + \frac{b}{a^2} + \frac{b^2}{a^3} + \ldots.$$

Is it true that the sum of this infinite series equals $1/(a - b)$?

Probably you know the answer, and even how to prove it, but let me recall. This series is a special case of the infinite *geometric progression*, or geometric series. In general, an infinite geometric progression is a series of the form

$$p + pq + pq^2 + pq^3 + \ldots.$$

Suppose that this sum has a meaning, and indeed equals to some number S, and let us try to find it, assuming that we can manipulate infinite sums as if they were finite. Then we can write

$$S = p + pq + pq^2 + \ldots = p + q(p + pq + pq^2 + \ldots) = p + qS,$$

from which we easily find

$$S = \frac{p}{1 - q}. \tag{1}$$

Testing this with Achilles and Turtle gives the correct result (check!). Does this convince you that the formula we derived is always correct?

Let us consider another example. Find the infinite sum

$$S = 1 - 1 + 1 - 1 + 1 - 1 + \ldots = \sum_{n=0}^{\infty} (-1)^n.$$

We can try the same method which was used for the geometric progression:

$$S = 1 - 1 + 1 - 1 + \ldots = 1 - (1 - 1 + 1 - 1 + \ldots) = 1 - S,$$

which gives $S = 1/2$. Is this a plausible result? Let us try to test it with our friends Achilles and the Turtle, by staging a different competition proposed by a Russian mathematician Evgeny Shchepin. Suppose first that Achilles and the Turtle have the same speed 1. Further, suppose that Achilles is blindfolded, and Zeno (or Shchepin) tells him in what direction to run. And the Turtle runs *towards* Achilles instead of running away from him, probably out of compassion to the blindfolded runner. The initial position is the same. Achilles runs as he was directed and in time 1 reaches the place where the Turtle was in the beginning. Meanwhile the turtle run with the same speed in the opposite direction, and reaches the starting point of Achilles. At this moment, Zeno cries: "Achilles, the Turtle is behind you!" Achilles turns around and in the next interval of time runs to his initial position. In the same time, the Turtle runs to her initial position. And so on. Now, what is the time when Achilles "passes the Turtle", that is meets her for the first time? Right, 1/2 !

As a third example, also due to Shchepin, suppose that Achilles speed is *less* than the speed of the Turtle. And they start running simultaneously in the same direction, the Turtle starting one unit ahead. We intuitively feel that in this case, Achilles will never reach the Turtle. What then our formula says? suppose for example that $a = 1$ and $b = 2$, then the formula

$$\frac{1}{a} + \frac{b}{a^2} + \frac{b^2}{a^3} + \ldots = \frac{1}{a-b}$$

says

$$1 + 2 + 4 + 8 + 16 + \ldots = -1.$$

Absurd? Maybe not. If we assume that they were running all the time with the same speed, not only after the start signal, but *also before it*, then they must have met at the time -1, that is one time unit before the start signal. Which is consistent with our formula. They never meet in the future, but they did meet in the past, before the start signal.

Let us consider the series from the second example, and group the terms differently:

$$S = 1 - 1 + 1 - 1 + \ldots = (1-1) + (1-1) + \ldots = 0 + 0 + 0 + \ldots$$

It looks like the sum is 0 at this time. Or, we can group in a still another way:

$$S = 1 - (1-1) - (1-1) - \ldots = 1.$$

These results contradict each other. This shows that when manipulating with infinite sums we cannot use the rules established for finite sums. This is why we need some general theory of summation of infinite series.

We will see that infinite series will play a very important role in this course. For an exposition of Calculus based on infinite series as the main object (and thus downplaying the role of such things as integrals and even limits and continuity) you may want to look at the book of Shchepin.

References

[1] Evgeny Shchepin, On Euler's footsteps,
http://www.pdmi.ras.ru/õlegviro/Shchepin/index.html