Infinite series II

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Isaac Newton is frequently called one of the most influential persons in history. And certainly he was the most influential physicist or mathematician of the Modern age[1].

Some of the main things Newton is credited with are: discovery of dissolution of the white light into colors, invention of the Newtonian telescope, the Universal law of gravitation, invention of Calculus, Newton's binomial formula, etc.

Here I am going to discuss the last two items. Newton did not publish anything on Calculus, but nevertheless in his older age he was involved in a bitter priority dispute with Leibniz about this.

One of the arguments used in this dispute was a letter Newton wrote earlier to the secretary of the Royal society (Henry Oldenburg) for transfer to Leibniz. The goal of such method of correspondence was apparently to be able to claim priority later; a copy of the letter was preserved in the archives of the Royal society.

In this letter he gives some hints about his inventions. This letter is interesting for us, because in it we can read how Newton himself describes his main discoveries, and which discoveries are most important on his own opinion.

If you think that is is the Newton-Leibniz formula you were taught in Calculus, you did not guess. It is all about *infinite series.*

As examples, the letter contains the series

$$\int_0^x \sqrt{1-t^2}dt = x - \frac{1}{2\cdot 3}x^3 - \frac{1}{5\cdot 8}x^5 - \frac{1}{7\cdot 16}x^7 - \frac{5}{9\cdot 128}x^9 + \dots, \qquad (1)$$

[1]Modern age is a usual name of the period in European history which started after the Middle age, that is approximately in XVI century. See Wikipedia article "Modern history" for a discussion of these terms.

and the binomial series

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots.$$

The binomial formula for $(a+b)^n$ with a positive integer n was known long time before Newton. It is this series which is proper to call Newton's binomial.

The letter also contains general formulations of Newton's discoveries in calculus, however they are encoded as anagrams[2]. The point was that he did not reveal his discoveries to Leibniz, but later could prove his priority by publishing the deciphering of the anagrams.

The two anagrams were these:

6accd ae 13eff7i3l9n4o4qrr4s8t12vx

which means:

Data aequatione quotcunque fluentes quantitates involvent fluxiones invenire et vice versa,

which is literally translated as:

Given an equation involving any number of fluent quantities, to find the fluxions, and vice versa.

In our modern language, fluent quantities are functions and fluxions are derivatives. The second anagram is longer and more interesting:

5accdæ10effh11i4l3m9n6oqqr8s11t9y3x:

11ab3cdd10eæg10ill4m7n6o3p3q6r5s11t8vx,

3acæ4egh5i4l4m5n8oq4r3s6t4v,

aaddæcecceiijmmnnooprrrsssssttuu.

which after deciphering and translating into English means:

One method consists in extracting a fluent quantity from an equation at the same time involving its fluxion; but another by assuming a series for any unknown quantity whatever, from which the rest could conveniently be derived, and in collecting homologous terms of the resulting equation in order to elicit the terms of the assumed series

First part of this sentence can be restated as: the method consists in solving differential equations. The second part says that *one can solve equations by substituting a series with undetermined coefficients and then determine the coefficients one-by-one.*

This is indeed a great discovery, and in what follows I will try to explain it with examples.

[2] An anagram is a coded message where the letters of the original message are permuted.

Example 1. Binomial series, or Newton's binomial.

$$(1+z)^\alpha = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!} z^n.$$

The left hand side means the principal branch which equals 1 at 0. The expansion holds in the disk of convergence $|z| < 1$, for any complex α. Check that when $\alpha = m$ is a positive integer then this formula is consistent with the usual binomial formula

$$(a+b)^m = \sum_{n=0}^{m} \frac{n!(m-n)!}{m!} a^n b^{m-n}.$$

The formula is proved by computing derivatives and applying Taylor's formula.

Exercise: use this formula with $\alpha = 1/2$ to obtain formula (1).

Example 2.

$$\mathrm{Log}\,(1+z) = z - z^2/2 + z^3/3 - z^4/4 + \ldots = \sum_{n=1}^{\infty} (-1)^{m-1} z^n/n.$$

Example 3. Evaluate the integral in the form of a series:

$$I = \int_0^1 \frac{\mathrm{Log}\,(1+x)}{x} dx.$$

First you should check that this integral is convergent. Since $\mathrm{Log}\,(1+z)$ equals 0 at $z = 0$, zero is a removable singularity, and the function is in fact continuous on $[0,1]$. Using the series in Example 2, and integrating term-by-term, we obtain

$$I = 1 - 1/4 + 1/9 - 1/36 + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}.$$

I will explain in a later lecture how to find the sum of this series explicitly. See also the text on Bernoulli numbers.

Example 4. Length of an ellipse. An ellipse can be described by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

A parametrization is obtained if we put

$$x = a\cos t, \quad y = b\sin t, \quad 0 \le t \le 2\pi.$$

The formula for the length of a curve from Calculus gives the length as

$$\int_0^{2\pi} \sqrt{(x')^2 + (y')^2} dt = \int_0^{2\pi} \sqrt{a^2\sin^2 t + b^2\cos^2 t} dt.$$

We may assume without loss of generality that $b \ge a$. The expression under the integral can be transformed as

$$\sqrt{b^2 - (b^2 - a^2)\sin^2 t} = b\sqrt{2 - e^2\sin^2 t},$$

where $e = \sqrt{b^2 - a^2}/b$ is called the eccentricity. The quantities b (the length of the larger semi-axis) and the eccentricity describe the size and the shape of the ellipse.

It is sufficient to find the length of the arc of the ellipse in the first quadrant, because the ellipse consists of four such arcs of equal lengths.

Thus we need to evaluate the integral

$$\int_0^{\pi/2} \sqrt{1 - e^2\sin^2 t} dt,$$

which gives the length of 1/4 of the ellipse whose larger semi-axis is 1.

There is no closed form expression using only elementary functions and constants related to them, like e and π. However there is the following series expansion:

$$\int_0^{\pi/2} \sqrt{1 - e^2\sin^2\theta}\, d\theta = \frac{\pi}{2}\left(1 - \frac{1}{2\cdot 2}e^2 - \frac{1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}e^4 - \frac{1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}e^6 - \dots\right)$$

$$= \frac{\pi}{2}\left(1 - \frac{1}{2\cdot 2}e^2 - \sum_{n=2}^{\infty} \frac{(2n-1)!!(2n-3)!!}{((2n)!!)^2} e^{2n}\right).$$

Sketch of the proof. First, denote $e^2\sin^2\theta$ by x and use the binomial formula:

$$(1-x)^{1/2} = 1 - \frac{1}{2}x - \sum_{n=2}^{\infty} \frac{(2n-3)!!}{2^n n!} x^n.$$

Second,

$$\int_0^{\pi/2} \sin^{2n}\theta\, d\theta = \frac{\pi}{2} 2^{-2n} \binom{2n}{n}.$$

The last integral is computed by the residues. This is one of our standard integrals. Do this!

Besides computing integrals, Newton claimed that he can "solve any differential equation". Probably he meant the following: Suppose that a differential equation with an initial condition is given

$$y^{(m)} = F(x, y, y', \ldots, y^{(m-1)}), \quad y(0) = c_0,\ y'(0) = c_1, \ldots, y^{(m-1)}(0) = c_{m-1}.$$

Then one can compute

$$y^{(m)}(0) = F(0, c_0, c_1, \ldots, c_{m-1}),$$

and all other derivative one-by-one by differentiating the equation, plugging $x = 0$ and using derivatives that are already computed. Then use Taylor's formula to obtain a solution in the form of a power series.

Example 5. Obtain the power series for $\sin z$ and $\cos z$ by solving the differential equation

$$y'' + y = 0$$

with initial conditions $y(0) = 0,\ y'(0) = 1$ for sine, and $y(0) = 1,\ y(0) = 0$ for cosine.

Example 6. Airy's differential equation Consider the following differential equation with initial conditions:

$$y'' = xy, \quad y(0) = 1,\ y'(0) = 0.$$

Its solutions are not elementary functions, so power series (or integrals) are essentially he only ways to represent solutions.

Substitute the series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \quad a_0 = 1, \quad a_1 = 0.$$

We obtain

$$y''(z) = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}.$$

The terms with $n = 0$ and $n = 1$ are absent, so we can write

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n,$$

where we “shifted the index”s, that is replaced n by $n+2$ in the sum. For the RHS we have

$$xy(x) = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n.$$

Now for the left and right hand sides to be equal, their coefficients have to be equal, so we obtain

$$(n+2)(n+1)a_{n+2} = a_{n-1}, \quad n = 1, 2, 3, \ldots.$$

Since $a_1 = 0$, we have $a_4 = a_7 = a_{10} = \ldots = 0$. Now RHS has no constant term, and we conclude that $a_2 = 0$. Then $a_5 = a_8 = a_{11} = \ldots = 0$. So the series contains only terms with x^n where n is divisible by 3. Then from the recurrent relation we find:

$$a_0 = 1, \; a_3 = \frac{1}{2.3}, \; a_6 = \frac{1}{2.3.5.6}, \; a_9 = \frac{1}{2.3.5.6.8.9}, \ldots.$$

Exercise. Obtain a power series solution of the same equation with initial conditions $y(0) = 0$, $y'(0) = 1$. Find the radii of convergence of both solutions.