High energy eigenfunctions of one-dimensional Schrödinger operators with polynomial potentials

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Abstract

For a class of one-dimensional Schrödinger operators with polynomial potentials that includes Hermitian and PT-symmetric operators, we show that the zeros of scaled eigenfunctions have a limit distribution in the complex plane as the eigenvalues tend to infinity. This limit distribution depends only on the degree of the polynomial potential and on the boundary conditions.

Keywords: *Eigenfunctions, PT-symmetry, Stokes phenomena, asymptotics.* MSC: 34B05, 34L20, 34M40, 34M60.

1. Introduction

We begin with an eigenvalue problem of the form

$$-y'' + P(x)y = \lambda y, \quad y(-\infty) = y(\infty) = 0, \tag{1}$$

where $P(x) = x^d + \ldots$ is a real monic polynomial of even degree d. The boundary condition is equivalent to $y \in L^2(\mathbf{R})$. It is well-known that the spectrum of this problem is discrete, all eigenvalues are real and simple, and they can be arranged into an increasing sequence $\lambda_0 < \lambda_1 < \ldots \to +\infty$. Moreover,

$$\lambda_n \sim \left(\frac{\pi d n}{2B(3/2, 1/d)}\right)^{\frac{2d}{d+2}} = \left(\frac{\sqrt{\pi}\Gamma(3/2 + 1/d)n}{\Gamma(1 + 1/d)}\right)^{\frac{2d}{d+2}}, \quad n \to \infty,$$

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where B and Γ are the Euler's functions. A general reference for these facts is [17].

Eigenfunctions y_n are entire functions of order $(d+2)/2$, and in this paper we study the distribution of their zeros in the complex plane when n is large. When $d = 2$ and P is even, we have $y_n(x) = H_n(x)\exp(-x^2/2)$, where H_n are the Hermite polynomials, and asymptotic distribution of zeros is known in this case in great detail [16]. The case $d = 4$, which is called sometimes an anharmonic oscillator or a double well potential, was studied much, but most attention was payed to the properties of eigenvalues, rather than the properties of eigenfunctions.

In [3], we proved that for $d = 4$ and even P, all zeros of all eigenfunctions belong to the union of the real and imaginary axis. The main result of the present paper implies that after an appropriate rescaling of the independent variable, zeros of eigenfunctions have a limit distribution in the plane, which depends only on d. This will be derived from the asymptotics of $\log|Y_n|$ as $n \to \infty$, where Y_n is a properly rescaled eigenfunction.

Let us consider real zeros first. According to the Sturm–Liouville theory, y_n has exactly n real zeros, all of them simple. A classical argument shows that all these real zeros belong to the interval (a_n^-, a_n^+) where a_n^- and a_n^+ are the smallest and the largest roots of the equation $P(x) - \lambda_n = 0$. The asymptotics

$$|a_n^\pm| \sim \lambda_n^{1/d}$$

suggests to define the rescaling:

$$Y_n(z) = y_n(\lambda_n^{1/d} z). \tag{2}$$

Now we denote by ν_n the counting measure of the roots of Y_n (for every set $X \subset \mathbf{C}$, $\nu_n(X)$ is the number of roots on Y_n in X). Our main result, Theorem 2 below, has the following corollary:

$$\frac{\nu_n|_{\mathbf{R}}}{n} \to c_d\sqrt{1-x^d}\,dx, \quad -1 \leq x \leq 1,$$

where $\nu_n|_{\mathbf{R}}$ are the restrictions of the measures on the real line, and the convergence is the usual weak convergence of measures: $\mu_n \to \mu$ means that $\int \phi d\mu_n \to \int \phi d\mu$ for each continuous function with compact support. The normalizing constant is

$$c_d = \frac{2}{d}B(3/2, 1/d).$$

For $d = 2$ the limit distribution is the "semi-circle law", the well-known asymptotic distribution of zeros of Hermite's polynomials.

A theorem of Hille [9, Theorem 11.3.3] implies that for every $r \in (0, 1)$ there exists $n_0(r) > 0$ such that for all $n > n_0$, all zeros of Y_n in the disc $\{z : |z| \leq r\}$ are real.

Zelditch [18] studies Laplace–Beltrami eigenfunctions on real analytic manifolds. Under certain conditions on the manifold, he extends the eigenfunctions to a complex neighborhood of the manifold and obtains a limit distribution of their zeros. Asymptotics in the complex domain help to study the distribution of real zeros. The same happens in our case.

Passing to the consideration of complex zeros, we make the same rescaling (2), and consider the rescaled counting measures

$$\mu_n = \nu_n / n \tag{3}$$

of zeros of Y_n in the complex plane.

Our main result will imply that these measures μ converge weakly to an explicitly described limit measure, which depends only on d.

Our results also apply to the so-called PT-symmetric Schrödinger operators which were intensively studied in the recent years [2, 13]. Let P be a complex polynomial of degree d with the property $P(-\overline{z}) = \overline{P(z)}$. Schrödinger operators with such potential P are called PT-symmetric. Every PT-symmetric potential can be written as $P(z) = P_1(iz)$, where P_1 is a polynomial with real coefficients. A real potential P is PT-symmetric if and only if P is even.

Following K. Shin [13], we consider the generalized eigenvalue problem which contains both self-adjoint and PT-symmetric cases.

$$-y'' + Py = \lambda y, \tag{4}$$

with

$$P(z) = (-1)^\ell (iz)^d + \sum_{k=1}^{d-1} a_k z^k,$$

where the coefficients a_k are arbitrary complex numbers, and the boundary condition is

$$y(re^{i\theta}) \to 0, \quad r \to \infty \quad \text{for} \quad \theta = -\frac{\pi}{2} \pm \frac{(\ell + 1)\pi}{d + 2}, \tag{5}$$

where $1 \le \ell \le d-1$. The self-adjoint problem (1) corresponds to the case that d is even, $\ell = d/2$, and a_k are real. The usual boundary condition imposed on a PT-symmetric potential corresponds to $\ell = 1$ [1, 2].

The following result belongs to Sibuya [14] and K. Shin [13].

Theorem A. *The spectrum of the problem* (4), (5) *is discrete, and to each eigenvalue corresponds a one-dimensional space of eigenfunctions. Eigenvalues satisfy*

$$\lambda_n \sim \left(\frac{\sqrt{\pi}\Gamma(3/2 + 1/d)n}{\sin(\ell\pi/d)\Gamma(1+1/d)} \right)^{\frac{2d}{d+2}}, \quad n \to \infty. \tag{6}$$

So we see from the asymptotics that the eigenvalues are "asymptotically real" in the sense that their arguments tend to zero. Shin proved that in the PT-symmetric case (that is when $a_k = (ib_k)^k$ with real b_k), all eigenvalues but finitely many are actually real.

Our main result, will imply that zeros of eigenfunctions y_n of the problem (4), (5), when properly rescaled as in (2) have a limit distribution that depends only on d and ℓ. The support of the limit distribution consists of some Stokes lines (which we later define precisely), and our result can be considered as a rigorous confirmation of the results of numerical computations of Bender, Boettcher and Savage [1].

Functions Y_n satisfy the differential equations of the form

$$Y_n'' = \lambda_n^{2/d}(P(\lambda_n^{1/d}z) - \lambda_n)Y_n = k_n^2((-1)^\ell(iz)^d - 1 + o(1))Y_n, \tag{7}$$

where $k_n = \lambda_n^{(d+2)/(2d)} \to \infty$, as $n \to \infty$. Here we choose the branch of $\lambda^{(d+2)/(2d)}$ which is positive on the positive ray.

It is well-known (see, for example [4, 7]) that the asymptotic behavior, as $n \to \infty$, of solutions of such differential equations depends on the quadratic differential

$$Q_{d,\ell}(z)dz^2 = ((-1)^\ell(iz)^d - 1)dz^2. \tag{8}$$

We recall the terminology and some known facts about such quadratic differentials. For the general theory of quadratic differentials we refer to [11, 15], and for applications to differential equations to [4, 5].

Let Q be an arbitrary polynomial. The zeros of Q are called the *turning points*. The curves on which $Q(z)dz^2 < 0$ are called the *(vertical) trajectories*. Each branch of the integral $\int \sqrt{Q}dz$ maps trajectories into vertical lines. The trajectories make a foliation of the plane with the singularities at the turning points. The leaves of this foliation are maximal smooth open curves on which

$Q(z)dz^2 < 0$ holds. Each leaf is homeomorphic to an open interval, and each end of a leaf is either at infinity or at a turning point. Those leaves which have at least one end at a turning point are called the *Stokes lines.* A Stokes line whose both ends are turning points is called *short.* At each simple zero of Q exactly three Stokes lines converge, and they make equal angles of $2\pi/3$ between them.

The components of the complement of the union of turning points and Stokes lines will be called the *Stokes regions.* These regions are unbounded and simply connected. The closure of each Stokes region (in $\mathbf{C}$) is homeomorphic either to a closed half-plane or to a closed strip. We say that these regions are *of the half-plane type* or *of the strip type*, respectively.

Turning points, Stokes lines and Stokes regions make a decomposition of the plane which we call the *Stokes complex.*

Our first result is the topological description of the Stokes complex[1] corresponding to the differential (8).

Theorem 1. *The Stokes complex of $Q_{d,\ell}dz^2$ is symmetric with respect to the imaginary axis and has the following property: Every turning point v which does not lie on the imaginary axis is connected by a short Stokes line with the turning point $-\overline{v}$ symmetric to v with respect to the imaginary axis.*

It is easy to see that this theorem yields a complete topological description of the Stokes complex. The turning points are the roots of the equation $(-1)^\ell(iz)^d = 1$. All these roots are simple, so three Stokes lines meet at each turning point.

If v is a turning point in the (open) right half-plane, then there is one short Stokes line from v to $-\overline{v}$. The other two Stokes lines originating at v go to infinity. Indeed, if one of those two were short, we would obtain a bounded Stokes region which is impossible. These two Stokes lines are contained in the right half-plane (otherwise they would intersect the symmetric Stokes lines originating at $-\overline{v}$ which is impossible.) The picture in the left half-plane is symmetric to the picture in the right half-plane.

If v belongs to the imaginary axis, that is $v = \pm i$, then all three Stokes lines originating from v are unbounded. One of them coincides with a ray of the imaginary axis $(\pm i, \pm i\infty)$ and the other two are symmetric with respect to the imaginary axis.

[1]Explicit equations of the Stokes lines can be written in terms of hypergeometric functions, but it is easier to study the topology of our Stokes complex by qualitative methods.

See Figures 1–9 where the Stokes complexes of $Q_{d,\ell}dz^2$ for $d = 2, 3, 4$ and 6 and various values of ℓ are represented.

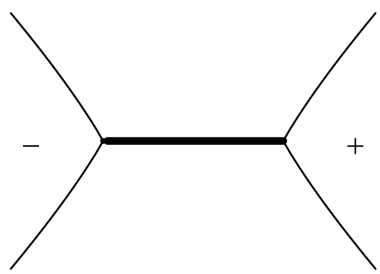

Fig. 1. $d = 2,\ \ell = 1$.

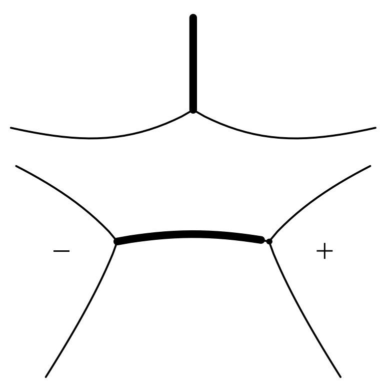

Fig. 2. $d = 3,\ \ell = 1$.

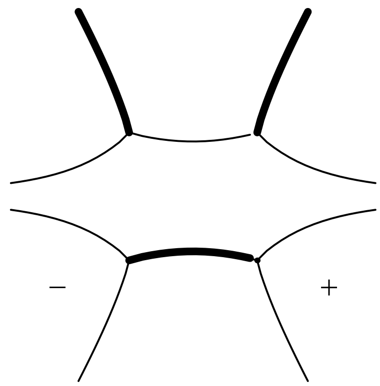

Fig. 3. $d = 4,\ \ell = 1.$

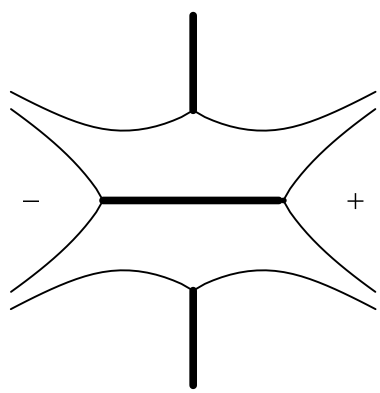

Fig. 4. $d = 4,\ \ell = 2.$

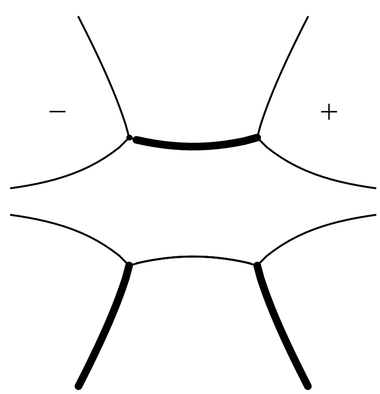

Fig. 5. $d = 4,\ \ell = 3.$

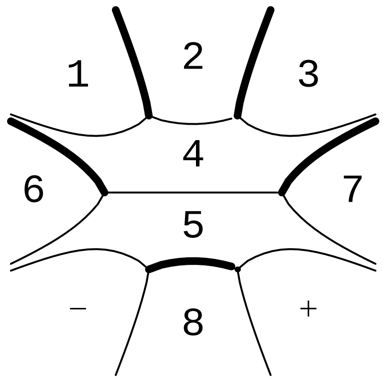

Fig. 6. $d = 6,\ \ell = 1$.

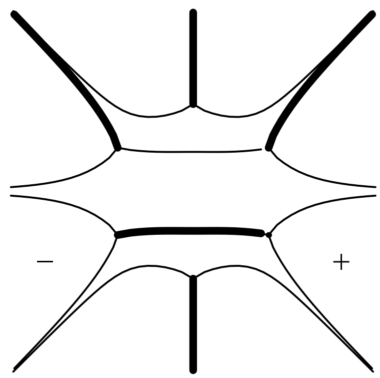

Fig. 7. $d = 6,\ \ell = 2$.

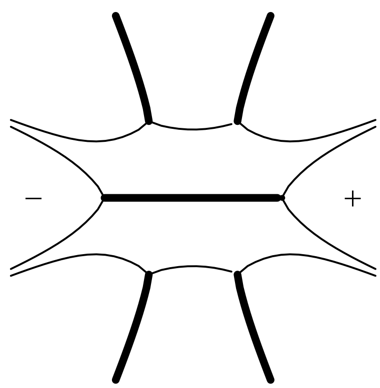

Fig. 8. $d = 6,\ \ell = 3$.

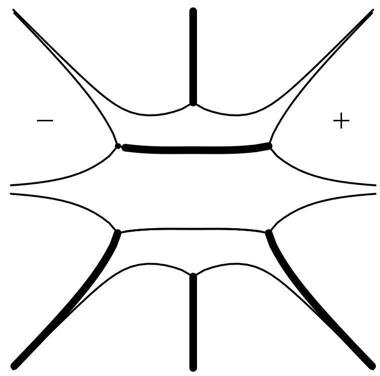

Fig. 9. $d = 6,\ \ell = 4.$

It is easy to see that each Stokes complex has exactly $d+2$ Stokes regions of the half-plane type. These regions are asymptotic to the sectors which are bounded by the *Stokes directions*

$$\{e^{i\theta} : \mathrm{Re} \int_0^{\exp(i\theta)} \sqrt{(-1)^\ell (it)^d} dt = 0\} \quad \text{that is} \quad \theta = -\frac{\pi}{2} + \frac{\pi(l+2k)}{d+2}.$$

The bisectors of the angles between adjacent Stokes directions are called the *anti-Stokes directions.* Each of the $d+2$ anti-Stokes directions approximately bisects some Stokes region of the half-plane type. Thus the boundary condition (5) indicates that $y(z) \to 0$ as $z \to \infty$ on two anti-Stokes directions symmetric with respect to the imaginary axis. This boundary condition singles out two Stokes regions of the type of half-plane: the right region ω^+ and the left region ω^-, so that the eigenfunction tends to zero along the bisectors of ω^- and ω^+. In Figures 1–9 the regions ω^+ and ω^- are marked by + and − signs, respectively. Figures 1, 4 and 8 correspond to self-adjoint problems.

It follows from the topological description of the Stokes complex of $Q_{d,\ell}dz^2$ above that the closures of ω^+ and ω^- are disjoint.

To state our main result, we need to define certain Stokes lines in the Stokes complex of $Q_{d,\ell}dz^2$ which we call *exceptional.* There will be exactly one exceptional Stokes line originating at each turning point.

Let v^+ and v^- be the turning points on the boundaries of ω^+ and ω^-, respectively. Then v^+ and v^- do not belong to the imaginary axis, so Theorem 1 implies that there is a short Stokes line E_0 from v^- to v^+. This short Stokes line is exceptional. For example, for the self-adjoint problem (1) we have $E_0 = (-1, 1)$.

If $v = i$ or $v = -i$ is a turning point, then the Stokes line $(i, +\infty i)$ or $(-i, -\infty i)$ is exceptional.

For the rest of turning points, exceptional Stokes lines are defined as follows. The complement of the set $\overline{\omega^+ \cup \omega^-} \cup E_0$, where the bar stands for the closure, consists of two components, we call these components D^+ (containing the positive imaginary ray) and D^- (containing the negative imaginary ray). Let v be a turning point in D^+ and not on the imaginary axis. Let L' and L'' be the two unbounded Stokes lines originating at v. Of these two Stokes lines we choose that one which lies between the other one and the positive imaginary axis, and call this chosen line exceptional. (Any family of disjoint curves tending to infinity in D^+ can be linearly ordered, for example anticlockwise; we used the word "between" in the sense of this order). Similarly, if v is a turning point in D^- and not on the imaginary axis, then of the two unbounded Stokes lines originating from v, the exceptional one is that which lies between the other one and the negative imaginary axis.

Exceptional Stokes lines are shown as bold lines in our figures.

Let E be the union of all exceptional Stokes lines and all turning points. We call E the exceptional set and denote

$$\Omega = \mathbf{C} \backslash E.$$

Then Ω is a doubly connected region, and $\sqrt{(-1)^\ell (iz)^d - 1}$ has two single-valued holomorphic branches in Ω. Consider the harmonic function in Ω

$$u(z) = \mathrm{Re} \int_0^z \sqrt{(-1)^\ell (it)^d - 1}\,dt = \mathrm{Re} \int_0^z \sqrt{Q_{d,\ell}(t)}\,dt. \tag{9}$$

Here we choose the branch of the $\sqrt{Q_{d,\ell}}$ in such a way that $u(z) \to -\infty$ as $z \to \infty$ along the anti-Stokes directions in ω^+ and ω^-. (The normalization (5) and the top coefficient of the potential P in (4) are chosen to make the choice of such branch of $\sqrt{Q_{d,\ell}}$ possible.)

The integral in the definition of u has one period corresponding to loops in Ω around the short Stokes line $E_0 = (v^-, v^+)$, but this period is pure imaginary, because $Q_{d,\ell} dz^2 < 0$ on E_0, thus the real part of this integral is a well defined harmonic function in Ω. In fact u is continuous and subharmonic in the whole plane (that the jumps of the integral on the exceptional Stokes lines are pure imaginary can be seen from the very definition of the Stokes lines). Now we can state our main result.

Theorem 2. *Let y_n be an eigenfunction of the problem* (4), (5) *corresponding to the eigenvalue λ_n, and normalized so that $|y_n(0)|+|y'_n(0)| = 1$. Put $Y_n(z) = y_n(\lambda_n^{1/d}z)$. Then*

$$\frac{1}{n}\log|Y_n| \to c_{d,\ell}u(z), \quad n \to \infty,$$

uniformly on compact subsets of Ω, and also in the sense of Schwartz distributions D' in **C**. *Here u is the function defined in* (9), *and*

$$c_{d,\ell} = \frac{\sqrt{\pi}\Gamma(3/2 + 1/d)}{\sin(\ell\pi/d)\Gamma(1 + 1/d)}.$$

Uniform convergence implies that the functions Y_n have no zeros on compact subsets $K \subset \Omega$ when $n > n_0(K)$. This can be made more precise by replacing a compact set by an unbounded subset of Ω which we describe below as an "admissible set". Each compact subset of Ω is contained in some admissible set.

The Laplace operator is continuous in D', and the Riesz measure $\mu = (2\pi)^{-1}\Delta u$ can be easily explicitly computed:

Corollary. *The normalized counting measures $\mu_n = \nu_n/n$ of the zeros of Y_n converge weakly to the measure*

$$\mu = c_{d,\ell}\sqrt{|Q_{d,\ell}(z)|}\,|dz|,$$

supported on E.

In particular, in the self-adjoint case (1) the limit density on $(-1, 1)$ is given by the formula $c_d\sqrt{1 - x^d}$, as advertised. Paper [1] contains nice pictures showing the zeros of the rescaled eigenfunctions of PT-symmetric operators clustering around E_0.

Recently a preprint of Hezari was published [8] where similar methods are used to study the self-adjoint eigenvalue problem

$$y'' = \lambda^2 q(x)y,$$

with a real polynomial q, $q(x) \to +\infty$ as $|x| \to \infty$ and with the usual boundary conditions on the real axis. Hezari shows that possible limit distributions of zeros of eigenfunctions y_n are supported by some vertical trajectories similar to our "exceptional set". However, in his setting, a single limit distribution as $n \to 0$ might not exist, as he shows by an example.

The main reason why the limit distribution exists in our case is Theorem 1, which describes the topology of a very special Stokes complex, while in Hezari's work an arbitrary Stokes complex symmetric with respect to the real line is involved.

2. Proofs

Proof of Theorem 1. It is convenient to make the change of the independent variable $z \mapsto iz$ which reduces our quadratic differential to the form $(\pm z^d + 1)dz^2$. It is clear that the Stokes complex of this new differential is symmetric with respect to the real line. All we have to prove is that there is a short Stokes line from every turning point in the upper half-plane to the complex conjugate turning point. The proof is performed in several steps.

Step 1. Let $P(z) = z^d + 1$. We prove that the Stokes complex of $P(z)dz^2$ contains a short Stokes line between the turning points $v_1 = e^{i\pi/d}$ and $v_{-1} = \overline{v_1} = e^{-i\pi/d}$. To find the directions of the Stokes lines originating at v_1, we write $P(z) = dv_1^{d-1}(z - v_1) + o(z - v_1)$, and

$$\int_{v_1}^{z} \sqrt{P(t)}dt = \frac{2}{3}\sqrt{d}v_1^{(d-1)/2}(z - v_1)^{3/2} + o(z - v_1)^{3/2}.$$

So on the tangents to the Stokes lines meeting at v_1 we have $(z-v_1)^3 v_1^{d-1} < 0$. Since $v_1^{d-1} = -1/v_1$, this is equivalent to

$$(z - v_1)^3 \in v_1\mathbf{R}_+. \tag{10}$$

Consider the sector S bounded by the segments $[0, 1]$, $[0, v_1]$ and the arc of the unit circle $[1, v_1]$. Our polynomial P maps S into the first quadrant, thus for $z \in S$, we have $\arg(z^d + 1) \in (0, \pi/2)$. This means that the direction ϕ of the vertical foliation $P(z)dz^2 < 0$ in S satisfies

$$\pi/4 < \phi + \pi k < \pi/2. \tag{11}$$

There is one Stokes line originating at v_1 with the direction in this interval. According to (10), its direction at v_1 is $\pi/(3d) - 2\pi/3$. Because of the limitations (11) on the direction of this line in S, it can leave S only through the interval $(0, 1)$. On this interval it has to meet the symmetric Stokes line from v_{-1}. Thus v_1 and v_{-1} are connected with a short Stokes line.

Step 2. The vertical foliation of the differential $P_\theta(z)dz^2 = e^{-2i\theta}P(e^{-i\theta}z)dz^2$ is obtained from the vertical foliation of $P(z)dz^2$ by counterclockwise rotation by θ. In particular, the Stokes complex of $P_\theta(z)dz^2$ contains a short Stokes line between the adjacent turning points $e^{i(\theta\pm\pi/d)}$.

Step 3. We prove that every turning point $v_k = e^{\pi i(2k-1)/d}$ of $P(z)dz^2$ in the (open) first quadrant is connected to the symmetric turning point $\overline{v_k}$ by a short Stokes line. This we prove by induction in k. For $k = 1$ the statement was proved in Step 1. Suppose it holds for $k \leq m - 1$, and let L_{m-1} be the Stokes line from v_{m-1} to its conjugate.

The tangents to the Stokes lines originating at v_m satisfy the equation $(z - v_m)^3 \in v_m\mathbf{R}_+$ similar to (10). In particular there is one Stokes line L_m starting at v_m in the direction $\phi_m = \pi(2m - 1)/(3d) - 2\pi/3$, so that $-2\pi/3 < \phi_m < -\pi/2$.

From Step 2 applied with $\theta = 2\pi(m - 1)/d$, there is a Stokes line U_m for the differential $P_\theta dz^2$ connecting v_m and v_{m-1}. Its tangent at v_m has direction

$$\psi_m = \phi_1 + \theta = \pi/(3d) - 2\pi/3 + 2\pi(m - 1)/d > \phi_d.$$

We are going to show that L_m never intersects U_m in the open unit disc. Proving this by contradiction, we suppose that z is a point of intersection, and denote by L'_m and U'_m the pieces of L_m and U_m from v_m to z. Then the integrals

$$\int_{L'_m} \sqrt{P(z)}dz \quad \text{and} \quad \int_{U'_m} \sqrt{P_\theta(z)}dz$$

are both non-zero (because the imaginary part of such integral is monotone along a Stokes line) and both pure imaginary. But $P_\theta = e^{-4\pi i(m-1)/d}P$ for our choice of θ, so we conclude that

$$\int_{L'_m} \sqrt{P(z)}dz \neq \int_{U'_m} \sqrt{P(z)}dz,$$

as these integrals have different arguments and are non-zero. This is impossible because the integral of $\sqrt{P}dz$ should be zero over any closed curve in the unit disc. Thus L_m and U_m do not intersect in the open unit disc.

For all z in the unit disc, the direction of the vertical foliation of $P(z)dz^2$ belongs to the interval $\pi/4 < \phi + \pi k < 3\pi/4$. Hence L_m cannot leave the upper half of the unit disc through the arc $[v_m, -1]$ of the unit circle.

Moreover, L_m cannot intersect the Stokes line L_{m-1} which exists by the induction assumption.

This leaves only one possibility that L_m leaves the upper half of the unit disc through the real axis. Then, by symmetry, it should continue to v_{-m}.

Step 4. Now we prove the same for the differential $P_-(z)dz^2 = (-z^d + 1)dz^2$, namely that every turning point in the first quadrant is connected by a short Stokes line with its conjugate. First we prove that the Stokes line L_1 starting at the turning point $z_1 = e^{2\pi i/d}$ in the direction $2\pi/(3d) - 2\pi/3$ cannot intersect the Stokes line of $P_{\pi/d}dz^2 = e^{-2\pi/d}P(z)dz^2$ connecting z_1 with 1, hence the only possibility for L_1 to leave the upper half of the unit disc is through the real line, so it should proceed to $\overline{z_1}$ by the symmetry. Then we proceed by induction on m to prove that there are Stokes lines for P_-dz^2 connecting the turning points z_m in the first quadrant with their complex conjugates. The induction step is similar to the one in Step 3 and we leave it to the reader.

Step 5. That every turning point in the left half-plane is connected by a short Stokes line with its complex conjugate is proved by the change of the independent variable $z \mapsto -z$.

This completes the proof of Theorem 1.

Proceeding to the proof of Theorem 2 we begin with some rough estimates.

Lemma 1. *Consider a normalized solution of the differential equation*

$$y'' = h^2 Qy, \quad y(0) = y_0, \quad y'(0) = y_1,$$

where $h > 0$ and Q is a holomorphic function satisfying $|Q(z)| \leq M$ for $|z| \leq R$. Then we have

$$|y(z)| \leq \max\{|y_0|, |y_1|\} \exp(hMR), \quad |z| < R.$$

Proof. We put $w_1 = hy$ and $w_2 = y'$, to obtain a matrix equation

$$\mathbf{w}'(z) = A(z)\mathbf{w}(z),$$

where

$$\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & h \\ hQ & 0 \end{pmatrix}.$$

This implies

$$\|\mathbf{w}(z)\| \leq \|\mathbf{w}(0)\| + \int_0^z \|A(t)\|\|\mathbf{w}(t)\|dt,$$

where we use the sup-norms

$$\|\mathbf{w}\| = \max\{|w_1|, |w_2|\},$$

and

$$\|A\| = \max\{|a_{11}| + |a_{12}|,\ |a_{21}| + |a_{22}|\} \leq hM.$$

Applying Gronwall's lemma [9, Thm. 1.6.6], we obtain the result.

Applying this result to the equation (7), and taking into account the asymptotics of the eigenvalues from Theorem A, we conclude that if the sequences $(1/n)\log|Y_n(z_0)|$ and $(1/n)\log|Y_n'(z_0)|$ are bounded at some point $z_0 \in \mathbf{C}$, then in every disc $|z| < R$ the family of subharmonic functions

$$u_n = \frac{1}{n}\log|Y_n| \tag{12}$$

is uniformly bounded from above. By a well-known theorem (see, for example, [10, Theorem 4.1.9]) it follows that from every sequence of these functions one can select a subsequence which either converges uniformly on every disc to $-\infty$, or converges in $D'(\mathbf{C})$ (or in $L^1_{\text{loc}}(\mathbf{C})$ which is equivalent for such families of subharmonic functions) to a subharmonic function u. Convergence to $-\infty$ is excluded by normalization of our eigenfunctions.

Thus the limit functions of our family are subharmonic, and the same theorem [10, Theorem 4.1.9] says that the Riesz measures of u_n weakly converge to the Riesz measure of u.

Thus all we need to do is to identify the possible limit functions u, and to show that there is only one. This will be done by proving the uniform convergence part of Theorem 2. Notice that subharmonic functions are upper semi-continuous, so the restriction of u on Ω defines u uniquely in the whole plane.

To prove the uniform convergence in Theorem 2, we need a finite open covering of Ω with the so-called canonical regions [4].

Consider a quadratic differential $Q(z)dz^2$ with arbitrary polynomial Q. The multi-valued function

$$\zeta(z) = \int^z \sqrt{Q(t)}dt \tag{13}$$

has holomorphic branches in each Stokes region. These branches are defined up to post-composition with a transformation $\zeta \mapsto \pm\zeta + c$, where c is an arbitrary constant. Each such branch maps its Stokes region univalently onto some right or left half-plane, or onto a vertical strip, depending on the type of the region [4, 5].

A region D, which is a union of Stokes regions and Stokes lines is called a *canonical region* if there is a holomorphic branch of ζ in D that maps D onto the plane with finitely many slits along some vertical rays.

It is clear that a canonical region contains exactly two distinct Stokes regions of the half-plane type. It is easy to state a criterion for a pair of Stokes regions of the half-plane type to belong to some canonical region:

Stokes regions D_1 and D_2 belong to a canonical region if and only if there is a curve from D_1 to D_2 that does not pass through the turning points, intersects the Stokes lines transversally, and intersects each component of the 1-skeleton of the Stokes complex at most once.

For example, in Figure 6 there is a canonical region containing the Stokes regions $-, 5, 6$, another one containing $-, 5, 4, 1$ and so on, but there is no canonical region containing $-$ and $+$.

It is easy to conclude from Theorem 1 that in the case that $Q = Q_{d,\ell}$, for each Stokes region D, other than ω^-, there is a canonical region containing ω^+ and D. Similar statement holds with ω^- and ω^+ interchanged. Moreover, for $Q = Q_{d,\ell}$, canonical regions containing ω^+ cover $\Omega\backslash\omega^-$, and canonical regions containing ω^- cover $\Omega\backslash\omega^+$. So the it is enough to prove the statement about uniform convergence in Theorem 2 for all canonical regions containing ω^+ or ω^-.

Let $Q(z)dz^2$ be a quadratic differential with a polynomial Q, and let a number $s > 0$ be given. A smooth curve $\gamma : (-1, 1) \to \mathbf{C}$ will be called s-admissible if it has the following properties:

(i) $\gamma(t) \to \infty$, as $|t| \to 1$,

(ii) For every $t \in (-1, 1)$, the (smaller) angle between $\gamma'(t)$ and the vertical trajectory at $\gamma(t)$ is at least s.

(iii) For every $t \in (-1, 1)$, the distance from $\gamma(t)$ to the set of turning points is at least s.

A subset $K \subset \Omega$, will be called s-admissible, if K is a union of s-admissible curves.

A subset $K \subset \Omega$ or a curve γ in Ω will be called simply *admissible* if it is s-admissible for some $s > 0$. It is easy to see that interiors of admissible subsets of canonical regions containing ω^+ or ω^- cover Ω. Two Stokes regions of the half-plane type belong to some canonical region if and only if there is an admissible curve passing through these two Stokes regions.

Now we have to study the behavior of the Stokes complex under small perturbations of the quadratic differential. For our purposes the following lemma will be sufficient.

Lemma 2. *Let $Q(z,h)$ be a polynomial of degree d with respect to z whose coefficients are continuous functions of h, for h in a neighborhood of 0, and suppose that $\deg_z Q(z,0) = d$, and that K is an admissible subset of some canonical region of $Q(z,0)dz^2$. Then there exists $h_0 > 0$, such that for $|h| < h_0$, the differential $Q(z,h)dz^2$ has a canonical region in which K is admissible.*

Proof. Let γ be an admissible curve for $Q(z,0)dz^2$. This means that γ does not pass through the zeros of $Q(z,0)$ and that the angles between the tangent lines to γ and the vertical trajectories of $Q(z,0)dz^2$ are bounded from below by some $s > 0$. It is clear that for every compact piece of γ these conditions persist for the vertical trajectories of $Q(z,h)dz^2$ for small h. Thus one only has to consider a neighborhood of ∞. Suppose that $Q(z,h) = a(h)z^d + \ldots$. Then there exist $h_0 > 0$ and $R > 0$ such that

$$|\arg Q(z,h) - d \arg z - \arg a(0)| < s/2,$$

whenever $|z| > R$ and $|h| < h_0$. The angle between γ' and the vertical direction is

$$\arg \gamma' - \frac{\pi}{2} + \frac{\arg Q}{2} + \pi k.$$

so if this angle is at least s for $h = 0$, then it is at least $s/2$ for $|h| < h_0$ and $|z| > R$. This proves the lemma.

The following lemma is essentially well-known, though we could not find a convenient reference in the literature, so we include a proof. This lemma constitutes the essence of the Liouville method, which is known to physicists as the WKB method (see [7, 12] for the general discussion of the method, including history). The closest statement to ours is the one in [4], but our proof is different, and it is based on Liouville's transformation.[2]

[2]We do not share the opinion of Fedoryuk [5, Ch. II, §1, sect. 6] that the Liouville's transformation is "less convenient" for complex functions than for the real ones.

Lemma 3. *Let Q be an arbitrary polynomial, and consider the differential equation*

$$-y'' + h^2 Q(z) y = 0, \tag{14}$$

where $h > 0$ is a parameter. Let D be a canonical region for $Q(z)dz^2$, containing a Stokes region D_1 of the half-plane type. Let $\zeta = \Phi(z)$ be a branch of

$$\int_{z_0}^{z} \sqrt{Q(w)} dw$$

which maps D onto the plane with vertical slits, and such that D_1 corresponds to a left half-plane, and let $K \subset D$ be an s-admissible set. Then, for each $h > h_0(s, Q)$, there exists a solution $y_h^*(z)$ of the differential equation (14)*, which satisfies*

$$y_h^*(z) = Q^{-1/4}(z) \exp(h\Phi(z))(1 + \epsilon(z, h)), \tag{15}$$

where

$$|\epsilon(z, h)| \leq \frac{h_0}{h - h_0}, \quad z \in K. \tag{16}$$

Moreover, one can take

$$h_0(s, Q) = \sup \int_\gamma \left| \frac{5}{16} \frac{Q'^2}{Q^3} - \frac{Q''}{4Q^2} \right| |d\zeta|, \tag{17}$$

where the sup *is over all s-admissible curves in K, and derivatives in* (17) *are with respect to z.*

Remark. It is easy to see that the integral in (17) is absolutely convergent. Assuming that the leading coefficient of Q has absolute value 1 and all roots of Q belong to the disc $|z| < 2$, we estimate this integral in terms of s. We have $|d\zeta| = \sqrt{|Q|}|dz|$, and thus the integrand is at most $A(s)(|\zeta| + 1)^{-(d+4)/(d+2)}$ on any s-admissible curve. Now, an s-admissible curve can be parametrized as $\Phi^{-1}(\xi + if(\xi)) : -\infty < \xi < \infty$, where f is a function whose derivative satisfies $|f'(\xi)| \leq C(s)$. So the integral does not exceed

$$A(s) \int_{-\infty}^{\infty} (|\xi| + 1)^{-(d+4)/(d+2)} \sqrt{1 + C^2(s)} d\xi.$$

So, the sup of these integrals defining h_0 is bounded by a constant which depends only on s, for all Q in a neighborhood of $Q_{d,\ell}$.

Proof of Lemma 3. We use the change of the variables which is due to Liouville (see, for example, [7]):

$$w(\zeta) = Q^{1/4}(\Phi^{-1}(\zeta))y(\Phi^{-1}(\zeta)),$$

Then the differential equation (14) becomes

$$w'' = h^2 w + gw, \tag{18}$$

where

$$g(\zeta) = \left(-\frac{5}{16}\frac{Q'^2}{Q^3} + \frac{Q''}{4Q^2}\right) \circ \Phi^{-1}(\zeta),$$

where the prime stands for the differentiation with respect to z.

Now we consider the integral equation

$$w(\zeta) = e^{h\zeta} + \frac{1}{2h}\int_{-\infty}^{\zeta} (e^{h(\zeta-t)} - e^{h(-\zeta+t)})g(t)w(t)dt, \tag{19}$$

where the path of integration γ is a part of the Φ-image of an s-admissible curve in D. It is easy to verify directly that every analytic solution of this integral equation satisfies (18). (To derive this integral equation one "solves" (18) by the method of variation of constants, considering gw as a given function, see [7]).

Putting in (19) $w = e^{h\zeta}W$, we obtain

$$W(\zeta) = 1 + \frac{1}{2h}\int_{-\infty}^{\zeta} (1 - e^{2h(t-\zeta)})g(t)W(t)dt. \tag{20}$$

This we solve by the method of successive approximation: set $W_0 = 0$ and define $W_{n+1} = 1 + F(W_n)$, where F is the integral operator in the right hand side of (20). Then we have for the sup-norms on γ:

$$\|W_{n+1} - W_n\|_\gamma \leq \frac{1}{2h}\max_{t\in\gamma}(1 + |e^{2h(t-\zeta)}|)\, \|W_n - W_{n-1}\|_\gamma \int_\gamma |g(t)|\, |dt|.$$

Property (ii) of admissible curves implies that γ intersects every vertical line at most once, so $\mathrm{Re}\,(t-\zeta) \leq 0$ on γ, and

$$|1 + e^{2h(t-\zeta)}| \leq 2,$$

Thus for $h > h_0$, the W_n's converge geometrically, uniformly on $\Phi(K)$ to a solution W of the integral equation (20). We have

$$W = W_1 + (W_2 - W_1) + (W_3 - W_2) + \ldots = 1 + \epsilon,$$

where ϵ satisfies (16). This proves the lemma.

Completion of the proof of Theorem 2. It remains to prove the statement about the uniform convergence. As admissible subsets of canonical regions cover Ω, it is enough to prove that the uniform convergence holds on each admissible subset. Let K be an s-admissible subset of a canonical region D containing ω^+ or ω^-. Our functions Y_n satisfy differential equations (7). For every sufficiently large n, we set $h_n = |k_n|$ and

$$Q_n^*(z) = \exp(2i \arg k_n)(P(\lambda_n^{1/d} z) - \lambda_n),$$

so that

$$Y_n'' = h_n^2 Q_n^* Y_n. \tag{21}$$

In view of (7) and $\arg k_n \to 0$, we conclude that $Q_n^* \to Q_{d,\ell}$ as $n \to \infty$ and all conditions of Lemma 2 are satisfied. When n is large enough, we apply Lemma 2 to Q_n^* and conclude that there exists a canonical region for $Q_n^* dz^2$ which contains K. According to the Remark after Lemma 3, there is a bound for h_0 in Lemma 3 that is independent of n. Applying Lemma 3 to this canonical region and equation (21), we obtain a solution y_n^* of the equation (21) which satisfies (15) with $h = h_n$ and some branch Φ_n of

$$\int^z \sqrt{Q_n^*(t)} dt.$$

In particular, this solution y_n^* tends to zero on the anti-Stokes direction in ω^+ or ω^-. Thus it is proportional to the eigenfunction. Our normalization of the eigenfunction implies that the coefficient of proportionality satisfies $\log |c_n| = O(n)$. So we obtain that

$$\frac{1}{h_n} \log |Y_n| \to u(z) + c,$$

uniformly on K, where c may depend on K. As the limit function exists and is continuous in the whole plane, and u is single-valued, we conclude that this last relation holds in the whole plane in the sense of D' with the same constant c. Comparing the values near 0 we conclude that $c = 0$.

This completes the proof.

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