Homogeneous potentials

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In the early 1990s, Bessis and Zinn-Justin experimentally discovered the surprising fact that all eigenvalues of the boundary value problem

$$-w'' + ix^3 w = Ew, \quad w(\pm\infty) = 0$$

are real. To explain this, Bender and Boettcher (1998) noticed that the potential $V(x) = ix^3$ has the following symmetry property $V(-\overline{x}) = \overline{V(x)}$, which is called PT-symmetry. All PT-symmetric problems have eigenvalues symmetric with respect to complex conjugation, but they are non necessarily real. To understand what happens in this case, Bender and Boettcher proposed to study one-parametric family of PT-symmetric equations boundary value problems

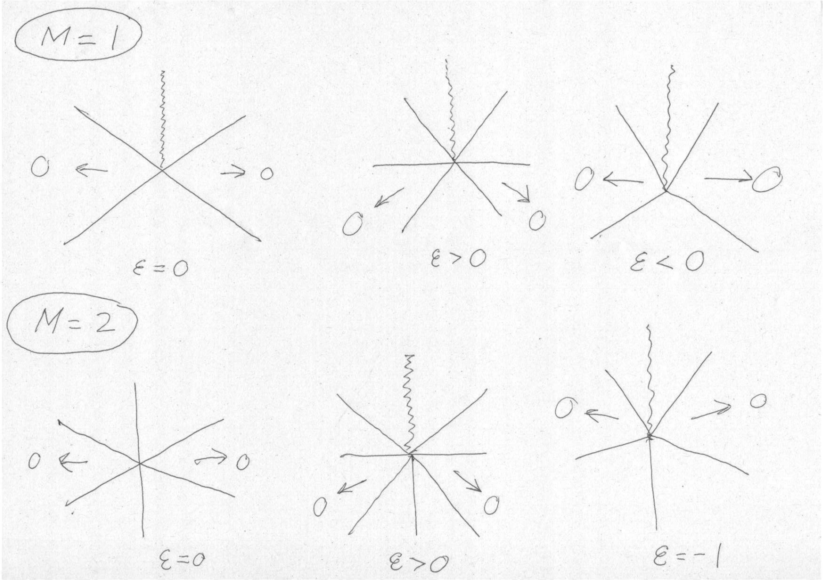
$$-w'' + x^{2M}(ix)^{\varepsilon} w = Ew, \quad w(x) \to 0, \quad x \to \infty, \quad x \in \Gamma.$$

Here ε and the principal branch is used, so the branch cut is the positive imaginary ray.

When the normalization contour is the real line, and $\varepsilon = 0$, the problem is self-adjoint and we have a sequence of real eigenvalues tending to $+\infty$. For sufficiently small $|\varepsilon|$ Bender and Bottcher observed interesting phenomena: all eigenvalues remain real when $\varepsilon > 0$ while only finitely many are real for $\varepsilon < 0$.
For larger values of $|\varepsilon|$ the normalization contour must be continuously deformed to ensure eigenvalues vary continuously with ε, as explained below.
Notice that:
$(M, \epsilon) = (1, 0)$ gives harmonic oscillator, $(M, \epsilon) = (2, 0)$ a quartic oscillator, both with real positive spectra, while
$(M, \epsilon) = (1, 1)$ is the cubic oscillator of Bessis and Zinn-Justin.
When $\epsilon = -M$ we have no eigenvalues at all.

The Stokes rays

are those rays in the complex plane on which $V(z)dz^2 < 0$, and the sectors between them are called the Stokes sectors. The theory of ODE says that for each Stokes sector S we have a one-dimensional family of solutions which tend to 0 exponentially on rays in S, while all other solutions grow exponentially on rays in S. Those which tend to zero in S are called subdominant in S. No non-trivial solution can be subdominant in adjacent sectors. For two non-adjacent sectors, the condition that there is a non-trivial solution subdominant in both of them can be considered as a boundary condition: such solution exists only when the spectral parameter E takes some discrete values. So we have a spectrum associated with any pair of non-adjacent Stokes sectors. The number of full sectors between these two normalization sectors is called the *level* of the PT-symmetric problem. If the normalization sectors contain the positive and negative rays, and $\varepsilon = 0$, the level equals M.



M=1
ε=0
ε>0
ε<0
M=2
ε=0
ε>0
ε=-1

When $\varepsilon_0 \neq 0$ we choose the normalization contour so that it tends to the real line by continuous deformation when ε changes from ε_0 to 0. This means that the level of our boundary value problem is always M.
We start with a problem of level M, $\varepsilon = 0$, $\Gamma = \mathbf{R}$. When ε increases, the contour Γ bends down, and the computation of Bender and Boettcher shows that all eigenvalues remain real.
When ε decreases, the contour Γ bends up, and almost all eigenvalues become non-real, *except* when ε is a negative integer $> -M$ at this moment they suddenly become all real again. When $\epsilon \to -M$, all eigenvalues tend to ∞ and disappear (Airy's equation).

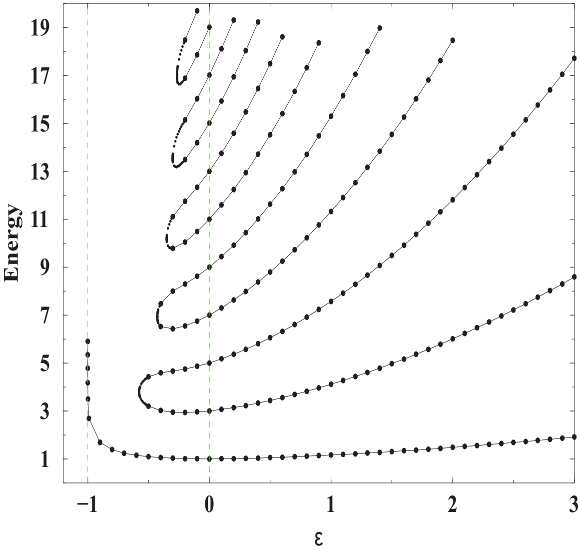


Figure: Case $M = 1$. Real eigenvalues as functions of ϵ.

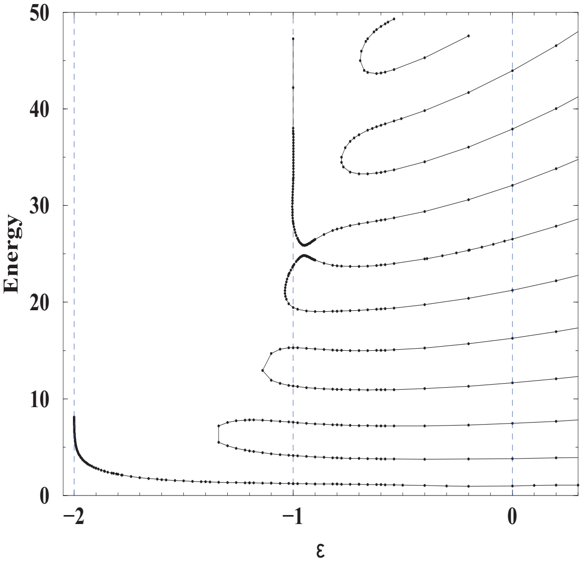


Figure: Case $M = 2$.

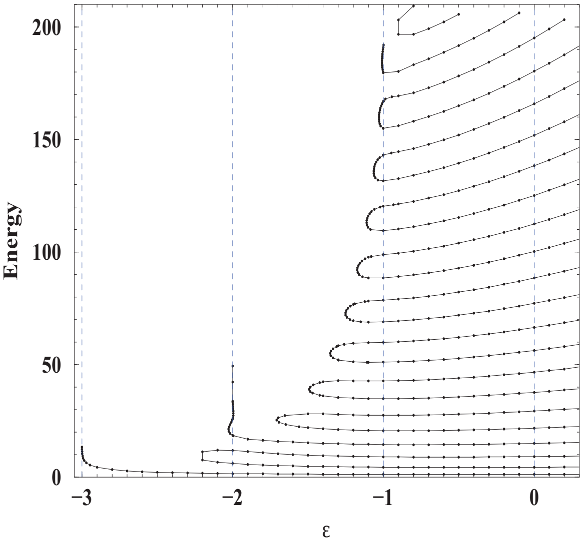


Figure: Case $M = 3$.

Known results.

1. When ε is an integer, and $M + \epsilon > 0$, all eigenvalues are real (K. Shin, 2005).
2. When $\varepsilon > 0$, all eigenvalues are real (Dorey, Dunning, Tateo, 2007)

Both results contain the conjecture of Bessis and Zinn-Justin ($(M, \varepsilon) = (1, 1)$), which was originally proved by Dorey, Dunning and Tateo (2001) and all known reality proofs are based on extension their method.
We will prove that all but finitely many eigenvalues are non-real when $\varepsilon \in (-M, 0)$ and ϵ is not an integer.

Theorem.

If $M = 1$, $\epsilon \in (-1, 0)$, almost all eigenvalues are non-real. Their arguments accumulate to

$$\pm\pi\frac{\epsilon}{4+\epsilon}.$$

If $M = 2$, $\epsilon \in (-2, 0)\backslash\{-1\}$, then almost all eigenvalues are non-real. Their arguments accumulate to

$$\pm\pi\frac{\epsilon}{6+\epsilon}.$$

Auxiliary self-adjoint problem

Eigenvalues of our problems are zeros of an entire function which we call the spectral determinant. (Usually there is a natural choice of this function). Let

$$m = 2M + \epsilon$$

and consider the self-adjoint problem

$$-y'' + (z^m + \lambda)y = 0, \tag{1}$$

with the boundary conditions $y(0) = y(+\infty) = 0$. This problem is self-adjoint and its spectral determinant is given by the following theorem which is essentially due to Sibuya.

Sibuya's Theorem.

For every λ there exists a unique solution y_0 of (1) satisfying

$$y_0(z,\lambda) = (1+o(1))z^{-m/4}\exp\left(-\frac{2}{m+2}z^{\frac{m+2}{2}} - \frac{\lambda}{2-m}z^{\frac{2-m}{2}}\right),$$

as $z \to +\infty$, $|\arg z| < 3\pi/(m+2) - \delta, \delta > 0$. The limit $f(\lambda) = \lim_{z\to 0+} y_0(z,\lambda)$ exists and is an *entire function* of λ. Its order is

$$\rho = \frac{1}{2} + \frac{1}{m}.$$

This entire function f is the spectral determinant of our self-adjoint problem: its zeros are exactly the negative of the the eigenvalues. So all these zeros lie on the negative ray.

It turns out that this function f contains complete information on all other boundary value problems of the type we consider (with homogeneous boundary conditions at ∞). More precisely, the spectral determinants of all these problems have expressions in terms of f.
For example, when $M = 1$ the spectral determinant $C(\lambda)$ of the PT-symmetric problem is

$$C(\lambda) = \frac{\omega^{1/2} f(\omega^2 \lambda) + \omega^{-1/2} f(\omega^{-2}\lambda)}{f(\lambda)}. \tag{2}$$

The fact that this fraction is entire (that is that zeros of the numerator contain the zeros of the denominator) is a highly restrictive property, and under some mild additional conditions it determines f uniquely. This fact is the basis of A. Voros's method of finding eigenvalues of our (self-adjoint) problem.
Our main object is the zeros of C. To study them, we use an asymptotic expansion of f.

Theorem 2.

$$f(\lambda) = \lambda^{-1/4}\exp(K_m\lambda^\rho)\Phi(\lambda^\rho),$$

where

$$\Phi(\mu) = 1 + \sum_{n=1}^{[m]} c_n\mu^{-n} + \frac{1}{8}\Gamma(m+1)\mu^{-m} + O(\mu^{-\kappa}),$$

as $|\lambda| \to \infty,\ |\arg\lambda| < \pi - \delta,\ \delta > 0,\ \kappa > m.$

In fact there is a full asymptotic expansion in powers of $\mu = \lambda^\rho$, but these powers are from the set

$$\Lambda = \{k_1 + k_2 m :\ k_1, k_2 \in \mathbf{N}_{>0}\},$$

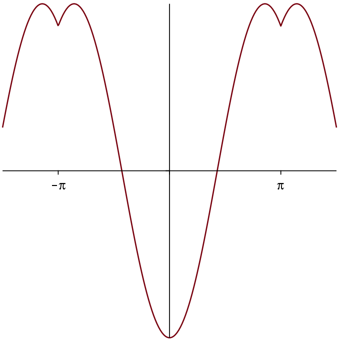
and the crucial fact for our argument is that the first non-integer power of μ in this expansion occurs with non-zero coefficient.

Derivation of Theorem 1 from Theorem 2

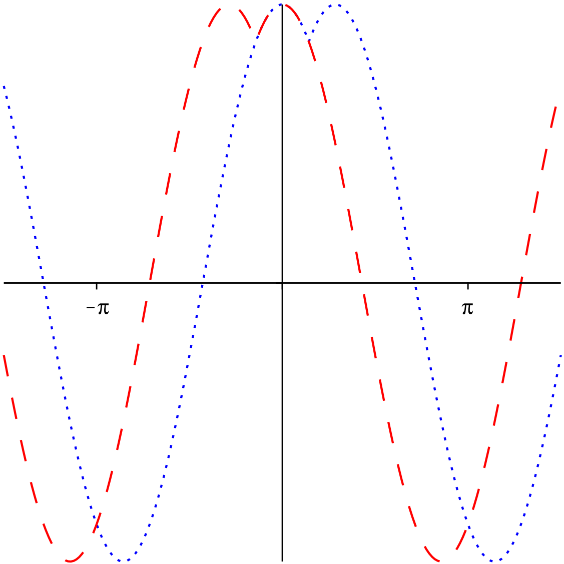
Indicator of an entire function f is defined by

$$h_f(\theta) = \limsup_{r\to\infty} \frac{\log|f(re^{i\theta})|}{r^\rho}.$$

For all functions considered here, the limit exists, except for finitely many θ. Indicator helps to visualize he principal term of the asymptotics. When $M = 1$, $\epsilon < 0$, $m \in (1, 2)$, $\rho \in (1, 3/2)$ the indicator of f looks like this:



Indicators of the summands in the numerator of C:



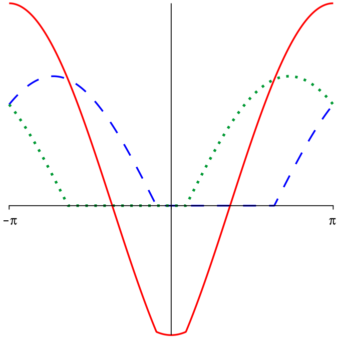
On the interval $|\theta| < \pi\epsilon/(4+\epsilon)$ indicators coincide, and we want to know whether a cancellation happens on this interval. Substituting our precise asymptotics from Theorem 2, we find that all terms with integer powers of λ^ρ cancel, but the first non-integer power does not. This shows that the summands have different asymptotics on this interval, so there are no zeros in the corresponding sector.
When ϵ is an integer, then *m* is also integer, and all terms of the asymptotics in the numerator cancel in our sector. In fact all zeros of *C* are positive according to Shin's result.

To show that there are only finitely many real zeros, it remains to consider what happens on the negative ray. The picture of indicators shows that there are zeros of the numerator there, and we can find their precise asymptotics, using the asymptotics of the summands. On the other hand, we also know the precise asymptotics of the zeros of the denominator,
and we know that the ratio is an entire function. This permits to conclude that *almost all* zeros of the numerator must cancel with the zeros of the denominator.
So the only rays to which the zeros of $C(\lambda)$ can accumulate are those stated in our theorem.

For $M = 2$, the formula for the spectral determinant is the following:

$$D(\lambda) = \frac{\omega f(\omega\lambda)f(\omega^3\lambda) + \omega^{-1}f(\omega^{-1}\lambda)f(\omega^{-3}\lambda) + f(\omega^3\lambda)f(\omega^{-3}\lambda)}{f(\omega\lambda)f(\omega^{-1}\lambda)}.$$

The indicators of the summands in the numerator are look like this:



Again we have an interval around $\theta = 0$ where two largest indicators are equal. The presence of a term with non-integer exponent in the asymptotics implies that there is no cancellation on this interval, and that the corresponding sector is free of eigenvalues. There are two other rays near which the numerator has zeros. But these zeros cancel with zeros of denominator, except finitely many.
We believe that similar arguments can be used for every M, but the calculation of spectral determinants and their indicators becomes more complicated.

Sketch of the proof of Theorem 2.

In the equation

$$y'' = (z^m + \lambda)y,$$

we make the change of the variables $w(z) = y(\lambda^{1/m}z)$ which reduces it to

$$w'' = \mu^2(z^m + 1)w, \quad \mu = \lambda^\rho.$$

Following Liouville, we change the independent variable to

$$\zeta = \Phi(z) = \int_0^z \sqrt{t^m + 1}dt,$$

and set

$$u(\zeta, \mu) = (z^m + 1)^{1/4}w(z, \mu), \quad z = \Phi^{-1}(\zeta).$$

This new function satisfies

$$u'' = \mu^2 u + gu, \quad \left(u'' = d^2u/d\zeta^2\right) \tag{3}$$

with some function

$$g(\zeta) = O(\zeta^{-2}), \quad \zeta \to \infty, \quad g(\zeta) = \frac{1}{4}m(m-1)\zeta^{m-2}, \quad \zeta \to 0.$$

Equation (3) is equivalent to the integral equation

$$u(\zeta,\mu) = e^{-\mu\zeta} + \frac{1}{\mu}\int_{\zeta}^{\infty} \sinh \mu(t-\zeta) g(t) u(t,\mu) dt.$$

Finally we set $F(\zeta,\mu) = e^{\mu\zeta} u(\zeta,\mu)$ and the integral equation simplifies to

$$F(\zeta,\mu) = 1 + \frac{1}{2\mu}\int_{\zeta}^{\infty} \left(1 - e^{\mu(\zeta-t)}\right) g(t) F(t,\mu) dt,$$

which is solved by the usual iteration method:

$$F = \mathbf{1} + \sum_{n=1}^{\infty} (2\mu)^{-n} \mathbf{T}_\mu^n[\mathbf{1}],$$

where

$$\mathbf{T}_\mu[F](\zeta) = \int_\zeta^\infty \left(1 - e^{\mu(\zeta - t)}\right) g(t) F(t) dt.$$

The principal term with non-integer exponent comes from

$$\mathbf{T}_\mu[\mathbf{1}](0) = \int_0^\infty g(t)dt + \int_0^\infty e^{-\mu t} g(t) dt.$$

First summand is constant while the second summand is

$$\frac{1}{4}\Gamma(m+1)\mu^{1-m} + O(\mu^{-m}).$$

It remains to deal with the rest of our iterated integrals, to make sure that all terms with non-integer exponents coming from them are smaller than those coming from the principal term.
These integrals are of the form

$$\begin{aligned}\mathbf{T}_\mu^n[\mathbf{1}](0) &= \int_0^\infty \left(1-e^{-t\mu}\right) g(t) \int_t^\infty \left(1-e^{\mu(t-t_1)}\right) g(t_1) \times \\ &\quad \int_{t_1}^\infty \left(1-e^{\mu(t_1-t_2)}\right) g(t_2)\ldots \\ &\quad \int_{t_{n-2}}^\infty \left(1-e^{\mu(t_{n-2}-t_{n-1})}\right) g(t_{n-1})\, dt_{n-1} dt_{n-2}\ldots dt_1 dt.\end{aligned}$$

Making the change of the variables

$$x_1 = t,\ x_2 = t_1 - t,\ x_3 = t_2 - t_1\ \ldots, x_n = t_{n-1} - t_{n-2}$$

and breaking out integral into 2^n summands, we obtain the integrals of the form

$$I_{n,J} = \int_0^\infty \ldots \int_0^\infty e^{-\mu L(\mathbf{x})} g(x_1) g(x_2) \ldots g(x_1 + \ldots + x_n)\, dx_n \ldots dx_1,$$

where $L(\mathbf{x}) = \sum_{j \in J} x_j$, and $J \subset \{1, \ldots, n\}$. Then we have the following

Lemma.

$$\mu^{-n} I_{n,J} = \sum_{k=0}^{[\alpha]+n+1} c_k \mu^{-k} + O(\mu^{-\alpha-1-n}).$$