

# Singularities of inverse functions

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## Abstract

A survey of general results on the singularities of inverses to meromorphic functions is given, with applications to holomorphic dynamics. This is a lecture delivered at the workshop “The role of complex analysis in complex dynamics” in Edinburgh on May 22 2013, with corrected mistakes and updated references.

## 1. Definition of singularities

Let  $f : D \rightarrow G$  be a non-constant holomorphic map between Riemann surfaces. Let  $z_0$  be a point in  $D$  such that  $f'(z_0) \neq 0$ . Then by the inverse function theorem there exists a neighborhood  $V$  of the point  $w_0 = f(z_0)$  and a holomorphic map  $\phi : V \rightarrow D$ , such that  $f \circ \phi = \text{id}_V$ .

What happens when we perform an analytic continuation of  $\phi$ ? Let  $\gamma : [0, 1] \rightarrow G$  be a curve from  $w_0$  to  $w_1$  and suppose that an analytic continuation of  $\phi$  along  $\gamma$  is possible for  $t \in [0, 1)$ , and let us see what can happen when  $t \rightarrow 1$ .

Consider the image  $\Gamma(t) = \phi(\gamma(t))$ . There are two possibilities:

a) The curve  $\Gamma(t)$  has a limit point  $z_1 \in D$  as  $t \rightarrow 1$ . By continuity we have  $f(z_1) = w_1$ . Now we conclude that the limit set of  $\Gamma(t)$  must consist of one point, because otherwise the limit set of the curve  $\Gamma$  would contain a continuum, while the preimage of a point under  $f$  is discrete. Thus  $\Gamma$  ends at  $z_1$ . If  $f'(z_1) \neq 0$ , then the analytic continuation of  $\phi$  to  $w_1$  is possible, and if  $f'(z_1) = 0$ , then  $\phi$  has an algebraic singularity (branch point) at  $w_1$ .

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b) The curve  $\Gamma(t)$  tends to  $\infty$ , where  $\infty$  is the added point of the one-point compactification of  $D$ . In this case  $\Gamma$  is an *asymptotic curve* of  $f$  which means that  $\Gamma$  is a curve in  $D$  parametrized by  $[0, 1)$ ,  $\Gamma(t) \rightarrow \infty$  as  $t \rightarrow 1$ , and  $f(\Gamma(t))$  has a limit in  $G$  as  $t \rightarrow 1$ . The limits of  $f$  along asymptotic curves are called *asymptotic values*.

Thus non-algebraic singularities of the inverse function  $f^{-1}$  correspond to asymptotic curves of  $f$ .

To obtain a one-to-one correspondence, we have to define precisely the notion of singularity, and to introduce some equivalence relation on the asymptotic curves. But first we notice the following:

**Proposition 1.** *If  $G$  contains no critical values and no asymptotic values, then  $f : D \rightarrow G$  is a covering map.*

We recall that a continuous map  $f$  is called a *covering* if every point  $w_1$  in the image has a neighborhood  $V$  such that every component of  $f^{-1}(V)$  is mapped onto  $V$  homeomorphically. An equivalent property is that every path in the image has a unique lifting. In our situation this is equivalent to saying that  $\phi$  can be analytically continued along any path in  $G$ .

Now we give an exact definition of a singularity of the inverse function. Let us assume that  $f(D)$  is dense in  $G$ . Let us fix a point  $a \in G$ . Every neighborhood  $V$  of  $a$  has non-empty open preimage. Consider a map  $S$  which to every neighborhood  $V$  of  $a$  puts into correspondence some component  $S(V)$  of  $f^{-1}(V)$ , so that the following condition is satisfied

$$V_1 \subset V_2 \longrightarrow S(V_1) \subset S(V_2).$$

Then there are two possibilities:

a) Intersection of all  $S(V)$  is not empty. Then it must consist of one point  $z \in D$  such that  $f(z) = a$ . Indeed, let this intersection contain a point  $z$ , and let  $U$  be a neighborhood of  $z$  then  $f(U)$  is a neighborhood of  $a$ , and there exists a neighborhood  $V$  of  $a$  such that  $\bar{V} \subset f(U)$ . This implies that  $S(V) \subset U$ . So  $\bigcap_V S(V)$  consists of one point  $z$ .

b) Intersection of all  $S(V)$  is empty. Then we say that our map  $S$  defines a *transcendental singularity* of  $f^{-1}$ . We say that the transcendental singularity  $S$  lies over  $a$ , and that  $a$  is the projection of the singularity  $S$ .

Introduction of transcendental singularities is a sort of completion of  $D$  to which our function  $f$  extends continuously. Let  $\text{Sing}(f)$  be the set of all transcendental singularities, and  $\bar{D}_f = D \cup \text{Sing}(f)$ , the disjoint union. We

define the topology on  $\overline{D}_f$  as follows: the neighborhoods of a point in  $D$  are its usual neighborhoods, and the neighborhoods of a point  $S \in \text{Sing}(f)$  are the sets  $S(V) \cup \{S\}$ . So the sets  $S(V)$  are *punctured neighborhoods* of  $S$ . We can define  $f(S) = a$  if  $S$  lies over  $a$ , and this extension of  $f$  to  $\overline{D}_f$  is continuous.

It is easy to see that for each transcendental singularity  $S$  over  $a$  there exists a curve  $\Gamma : [0, 1) \rightarrow D$  such that for every neighborhood  $V$  of  $a$  we have  $f(\Gamma(t)) \in V$  for all  $t$  sufficiently close to 1. Thus  $\Gamma$  is an asymptotic curve with asymptotic value  $a$ . And conversely, if  $\Gamma$  is an asymptotic curve with asymptotic value  $a$ , then we choose as  $S(V)$  to be that component of  $f^{-1}(V)$  which contains  $\Gamma(t)$  for  $t$  sufficiently close to 1, and this defines a transcendental singularity.

If  $S$  is a transcendental singularity, then all regions  $S(V)$  are unbounded. They are called *tracts* of  $f$  over  $a$ .

One can also define transcendental singularities as elements of the completion of  $D$  with respect to some metric. Suppose that  $G$  is equipped by some intrinsic metric  $\sigma$ . “Intrinsic” means that the distance between two points is equal to the infimum of the lengths of curves connecting these two points, for example, any smooth Riemannian metric is intrinsic. The pull-back  $\rho = f^*\sigma$  is an intrinsic metric in  $D$  defined as follows: the  $\rho$ -length of a curve in  $D$  is the  $\sigma$ -length of its image. So  $f$  becomes a local isometry  $(D, \rho) \rightarrow (G, \sigma)$ .

Now, we define another metric  $\rho_M$  in  $D$ , which is called the *Mazurkiewicz metric*. The  $\rho_M$ -distance between two points is the infimum of  $\sigma$ -diameters of curves  $f \circ \gamma$  over all curves  $\gamma$  connecting these two points. Mazurkiewicz’s metric is in general not intrinsic, and  $\rho_M \leq \rho$ . However  $\rho_M$  coincides with  $\rho$  on sufficiently small neighborhoods of a point  $z \in D$  for which  $f'(z) \neq 0$ .

**Example.** Let  $f(z) = \cos z$ ,  $\mathbf{C} \rightarrow \mathbf{C}$ . Let  $\sigma$  be the usual Euclidean metric. Then  $\rho = f^*\sigma$  is the metric whose line element is

$$|f'(z)||dz| = |\sin z||dz|.$$

The  $\rho$ -distance between the points 0 and  $2\pi m$  is  $2m$ , (the shortest curve is the segment  $[0, 2\pi m]$  which is mapped by  $f$  onto the segment  $[-1, 1]$  described  $2m$  times). The Mazurkiewicz distance between the same points is 2.

**Exercise.** Function  $f$  extends to a continuous function on the completion  $\overline{D}_M$  of  $D$  with respect to  $\rho_M$ , and that there is a homeomorphism  $\phi : \overline{D}_M \rightarrow$

$\overline{D}_f$ , such that the extension satisfies such that  $f(z) = f(\phi(z))$  for all  $z \in \overline{D}_M$ .

## 2. Iversen's classification

We call critical points and transcendental singularities considered as points of  $\overline{D}_f$  the singularities of  $f^{-1}$ .

We begin with the simplest kind of transcendental singularities, the isolated ones. Let  $S$  be an isolated transcendental singularity over a point  $a$ , then there is an open  $\sigma$ -disc  $V = B(a, r)$  of radius  $r$  around  $a$ , such that  $S(V)$  is at positive distance from other singularities. Proposition 1 applied to the restriction

$$f : S(V) \setminus f^{-1}(a) \rightarrow V \setminus \{a\} \quad (1)$$

implies that this restriction is a covering map. All possible coverings over a punctured disc are classified by subgroups of the fundamental group which is the infinite cyclic group. Thus there are two possibilities:

a) (1) is  $m$ -to-1, and  $S$  is a critical point, or

b) (1) is a universal cover. In this case  $S(V)$  is a simply connected region bounded by a single curve in  $D$ , parametrized by  $(0, 1)$ , and both ends of the curve are at  $\infty$ . The map (1) is equivalent to  $z \mapsto \exp(z)$  from the left half-plane to the punctured unit disc. This type of singularity is called *logarithmic*.

**Examples.** Function  $\exp : \mathbf{C} \rightarrow \mathbf{C}$  has one logarithmic singularity over 0. Function  $\exp : \mathbf{C} \rightarrow \overline{\mathbf{C}}$  has two logarithmic singularities, one over 0 another over  $\infty$ . Function  $\cos : \mathbf{C} \rightarrow \overline{\mathbf{C}}$  has two logarithmic singularities over  $\infty$  and infinitely many algebraic ones over 1 and  $-1$ . The entire function  $z \mapsto \sin z/z$ ,  $\mathbf{C} \rightarrow \overline{\mathbf{C}}$  has two logarithmic singularities over  $\infty$ , infinitely many critical points, and two non-isolated singularities over 0.

Further classification of transcendental singularities is due to Felix Iversen (1912).

A transcendental singularity  $S$  over  $a$  is called *direct* if there exists  $V$  such that  $f(z) \neq a$  for  $z \in S(V)$ . Otherwise it is called *indirect*.

So logarithmic singularities are direct.

**Examples.** We consider functions  $\mathbf{C} \rightarrow \overline{\mathbf{C}}$ . Function  $e^z \sin z$  has one direct non-logarithmic singularity over  $\infty$ . Function  $\sin z/z$  has two indirect singularities over 0, and function  $\sin \sqrt{z}/\sqrt{z}$  has one indirect singularity over 0.

### 3. Meromorphic functions of finite order

From now on we only consider maps  $\mathbf{C} \rightarrow \overline{\mathbf{C}}$ , and the Riemannian metric  $\sigma$  will be the spherical metric, whose pullback  $\rho = f^*\sigma$  has the length element

$$\frac{|f'(z)||dz|}{1 + |f(z)|^2}.$$

The lower order of growth of a meromorphic function will play an important role. It is defined by

$$\lambda(f) = \liminf_{r \rightarrow \infty} \frac{\log A(r, f)}{\log r},$$

where

$$A(r, f) = \frac{1}{\pi} \int_{|z| \leq r} \frac{|f'|^2}{(1 + |f|^2)^2} dm,$$

where  $dm$  is the Euclidean area element in the plane. The order of  $f$  is defined similarly, with  $\limsup$  instead of  $\liminf$ . The geometric interpretation of the quantity  $A(r, f)$  is the “average number of sheets” of the covering of  $\overline{\mathbf{C}}$  by the image of the disc  $|z| \leq r$ : the integral is the area of this image, and  $\pi$  is the area of  $\overline{\mathbf{C}}$ .

The Nevanlinna characteristic is defined by the averaging of  $A(r, f)$ ,

$$T(r, f) = \int_0^r A(t, f) \frac{dt}{t}.$$

It has an advantage that it satisfies the usual properties of the degree of a rational function,

$$T(r, f + g) \leq T(r, f) + T(r, g) + O(1), \quad T(r, fg) \leq T(r, f) + T(r, g) + O(1),$$

$$T(r, f^n) = nT(r, f) + O(1) \quad \text{and} \quad T(r, f) = T(r, 1/f).$$

The order and lower order can be defined using  $A(r, f)$ ,  $T(r, f)$  or,  $\log M(r, f)$  in the case of entire functions, with the same result.

**Theorem 1.** (Denjoy, Ahlfors, Beurling, Carleman). *The number of direct singularities of  $f^{-1}$  is at most  $\max\{1, 2\lambda(f)\}$ .*

**Corollary.** *If  $f$  is an entire function, then the number of transcendental singularities over points in  $\mathbf{C}$  is at most  $2\lambda$ . In particular, the number of finite asymptotic values of an entire function is at most  $2\lambda(f)$ .*

To derive the Corollary, one notices that the number of singularities of an entire function over infinity is at least the number of transcendental singularities over finite points. Indeed, between any two asymptotic curves corresponding to distinct transcendental singularities there must be an asymptotic curve with infinite asymptotic value. All singularities over infinity are direct.

There are several different proofs of Theorem 1, due to Ahlfors, Beurling and Carleman; each of them introduced new important tools of analysis.

The main result used in Carleman's proof of Theorem 1 is the following important inequality from potential theory. Let  $u$  be a non-negative subharmonic function in a ring,  $A = \{z : r_0 < |z| < r_1\}$ . Let

$$m_2(r) = \left( \int_0^{2\pi} u^2(re^{i\phi}) d\phi \right)^{1/2},$$

$$\mu(t) = m_2(e^t).$$

Then for  $r \in (r_0, r_1)$  we have

$$\mu''(t) \geq \left( \frac{\pi}{\theta(e^t)} \right)^2 \mu(t), \tag{2}$$

where  $\theta(r)$  is defined in the following way. If  $u(z) > 0$  on the circle  $\{z : |z| = r\}$  then  $\theta(r) = +\infty$ ; otherwise  $\theta(r)$  is the angular measure of the set  $\{\phi : u(re^{i\phi}) > 0\}$ .

Inequality (2) expresses a kind of convexity. If  $\theta \equiv +\infty$ , then this is the ordinary convexity of  $\mu(t)$ . Small  $\theta$  implies that  $\mu$  is "very convex".

It follows that when the set  $\{z \in A : u(z) > 0\}$  is narrow, the function  $u$  must grow fast. When  $er_0 < r_1 \leq +\infty$ , one can derive from (2) that

$$\log m_2(er) \geq \pi \int_{r_0}^r \frac{ds}{s\theta(s)} + \log \frac{dm_2}{d \log r} \Big|_{r=r_0}, \quad r_0 < r < r_1/e. \tag{3}$$

It is this version of Carleman's inequality that is most frequently used. When  $r_1 = \infty$ , inequality (3) is asymptotically best possible when  $r \rightarrow \infty$ ; extremal regions are angular sectors.

An appropriate version of Carleman's inequality holds in every dimension.

To derive Theorem 1 from Carleman's inequality we suppose without loss of generality that all direct singularities lie over finite points, and that there are at least two of them. Choose  $p \geq 2$  direct singularities of  $f^{-1}$  and consider Euclidean discs  $V_j, 1 \leq j \leq p$  of radii  $\epsilon$  around the corresponding asymptotic values  $a_j$ . We choose  $\epsilon$  so small that the sets  $D_j = S_j(V_j)$  are disjoint and  $f(z) \neq a_j$  in  $D_j$ . The last condition can be satisfied by definition of direct singularity.

Then we consider functions

$$u_j(z) = \log \frac{\epsilon}{|f(z) - a_j|}, \quad z \in D_j.$$

Each  $u_j$  is positive and harmonic in  $D_j$  and zero on the boundary. We denote

$$B_j(r) = \max_{|z|=r} u_j(z), \quad B(r) = \max_j B_j(r).$$

Then (3) implies for every  $j$

$$\log B(er) \geq \log m_2(er, u_j) \geq \pi \int_{r_0}^r \frac{ds}{s\theta_j(s)} + O(1)$$

Averaging in  $j$  gives

$$\log B(er) \geq \frac{\pi}{p} \int_{r_0}^r \frac{ds}{s} \sum_{j=1}^p \frac{1}{\theta_j(s)} + O(1). \quad (4)$$

On the other hand, by Cauchy-Schwarz inequality,

$$p^2 = \left( \sum_j \sqrt{\theta_j(s)} \frac{1}{\sqrt{\theta_j(s)}} \right)^2 \leq \sum_j \theta_j(s) \sum_j \frac{1}{\theta_j(s)} \leq 2\pi \sum_j \frac{1}{\theta_j(s)}.$$

Inserting this to (4), we obtain

$$\log B(er) \geq \frac{p}{2} \int_{r_0}^r \frac{ds}{s} + O(1) = \frac{p}{2} \log r + O(1).$$

By rudimentary Nevanlinna theory, this implies that

$$\liminf_{r \rightarrow \infty} r^{-p/2} T(r, f) > 0,$$

so the lower order of  $f$ , is at least  $p/2$ . When  $p = 1$  we do not obtain any estimate because many circles  $|z| = r$  may lie entirely in  $D_1$ .

**Theorem 2.** ([5, 12]) *For a meromorphic function  $f$  of finite lower order, each indirect singularity over a point  $a$  is a limit of critical points whose critical values are distinct from  $a$ .*

For functions of infinite order, this is not so; there are entire functions of infinite order without critical points at all, and such that the set  $\text{Sing}(f)$  has the power of continuum [16], and all but countably many singularities are indirect.

The main analytic tool in the proof of Theorem 2 is the Carleman inequality.

Theorem 2 helps to prove in many situations the existence of critical points. The simplest example is the following result, originally established by Clunie, Eremenko, Langley and Rossi:

**Theorem 3.** *Let  $f$  be a transcendental meromorphic function of order  $\rho$ .*

- a) If  $\rho < 1$ , then  $f'$  has infinitely many zeros,*
- b) If  $\rho < 1/2$ , then  $f'/f$  has infinitely many zeros,*
- c) If  $f$  is entire, and  $\rho < 1$ , then  $f'/f$  has infinitely many zeros.*

Here is another application of Theorem 2: *If  $f$  is a non-constant meromorphic function, then  $ff'$  takes every finite non-zero value.*

This was conjectured by Hayman in 1967, and all known proofs of this conjecture use Theorem 2<sup>1</sup>

#### 4. The sets of singularities of various types

The set of asymptotic values is an analytic (Suslin) set. This is a larger class than Borel sets. One of the several equivalent definitions is that a Suslin set is a continuous image of a Borel set.

More generally, the set of projections of singularities of an arbitrary multi-valued analytic function is an analytic set. This was proved by Mazurkiewicz who introduced his metric specially for this purpose.

Nothing more can be said, even if one considers asymptotic values of meromorphic functions of restricted growth.

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<sup>1</sup>This was written in 2013. It is shown in [1] that Hayman's conjecture also follows from a deep result of Yamanoi [17].



**Theorem 3.** (Cantón–Drasin–Granados) *For every analytic (Suslin) set  $A$ , and every  $\lambda \geq 0$  there exists a meromorphic function of order  $\lambda$  whose set of asymptotic values is equal to  $A$ .*

This is a difficult result which improves on two earlier simpler constructions:

*For every analytic set, there is an entire function whose set of asymptotic values is  $A \cup \{\infty\}$  (M. Heins),*

and

*For every  $\lambda \geq 0$ , there is a meromorphic function of order  $\lambda$  whose set of asymptotic values is  $\overline{\mathbf{C}}$ . (A. Eremenko).*

Thus there is no restriction on the size of the set of asymptotic values of a meromorphic function of given order, and the Corollary from Theorem 1 is the only restriction for entire functions of given order.

The asymptotic values coming from direct singularities are rare:

**Theorem 4.** (M. Heins) *Let  $f$  be a meromorphic function in  $\mathbf{C}$ ,  $V$  a disc, and  $D$  a component of  $f^{-1}(V)$ . Then the restriction  $f : D \rightarrow V$  takes every value in  $V$  with at most one exception. Thus the set of projections of direct singularities is at most countable.*

On the other hand, the set of direct singularities over one point can have the power of continuum [7].

If  $a$  is an omitted value of  $f$ , then there is at least one singularity over  $a$  (Iversen’s theorem). Evidently, all singularities over  $a$  are direct in this case, but it is possible that none of them is logarithmic.

Non-logarithmic direct singularities of inverses of *entire* functions have additional interesting properties:

**Theorem 5.** (Sixsmith) *Let  $a \in \overline{\mathbf{C}}$  be a projection of a direct non-logarithmic singularity of the inverse of an entire function. Then either  $a$  is a limit of critical values, or every neighborhood of this singularity contains another transcendental singularity which is either indirect or logarithmic and whose projection is different from  $a$ .*

**Theorem 6.** [7] *Let  $S$  be a direct non-logarithmic singularity of the inverse of an entire function over a point  $a \in \mathbf{C}$ . Then every neighborhood of  $S$  contains other direct singularities over the same point  $a$ .*

It follows that whenever the inverse of an entire function has a direct non-logarithmic singularity over a finite point, it must have the set of singularities

of the power of continuum over the same point.

Another corollary is that direct singularities over finite points of inverses of entire functions of finite order are all logarithmic.

## 5. Classes of functions defined by restrictions on their singular values

By *singular values* we mean critical and asymptotic values.

The simplest class is the Speiser class  $\mathbf{S}$  which consists of meromorphic functions with finitely many critical and asymptotic values. It is the union of classes  $\mathbf{S}_q$  which consist of functions with  $q$  critical and asymptotic values.

These are the simplest meromorphic functions from the geometric point of view. Examples are  $\exp(z)$ ,  $\cos z$ ,  $\wp(z)$ , rational functions. The class of entire functions in  $\mathbf{S}$  is closed under composition. To describe the most important property of functions of class  $\mathbf{S}$  we need a definition.

*Definition.* Two meromorphic functions  $f$  and  $g$  defined in simply connected regions are called topologically equivalent if there exist homeomorphisms  $\phi$  and  $\psi$  such that  $f \circ \phi = \psi \circ g$ .

**Theorem 7.** (Teichmüller, Eremenko–Lyubich) *Let  $f$  be a function of class  $\mathbf{S}$ , and  $M_f$  the set of meromorphic functions equivalent to  $f$ . Then all functions in  $M_f$  are defined in  $\mathbf{C}$ , and the set  $M_f$  is a complex analytic manifold of dimension  $q + 2$ , on which the critical and asymptotic values are holomorphic.*

These manifolds were introduced and studied in [10]. The crucial property here is that  $M_f$  has finite dimension. This permits to extend Sullivan’s proof of the absence of wandering domains to functions of class  $\mathbf{S}$ .

In the study of holomorphic families of entire functions, the dependence of periodic points on parameter is important. Let us fix some  $g \in \mathbf{S}$ , and let  $f \in M_g$ . Consider the equation for a periodic point

$$f^m(z) = z.$$

Solution of this equation  $z = \alpha(f)$  is a multi-valued function on  $M_g$ . The main result on this multi-valued function is

**Theorem 8.** (Eremenko–Lyubich) *The function  $\alpha$  has only algebraic singularities on  $M_g$ .*

The proof uses a version of Carleman’s inequality (2) which takes into account not only the “width” of  $D$  but also the amount of “spiraling” of  $D$  as  $z \rightarrow \infty$ .

Class  $\mathbf{B}$  consists of transcendental entire functions whose set of singular values is bounded. As  $\infty$  is always an asymptotic value of an entire function, it follows that for  $f \in \mathbf{B}$ , all singularities over  $\infty$  are isolated and thus logarithmic. This means that the behavior of the inverse  $f^{-1}$  near infinity is as simple as possible for a transcendental function. Evidently  $\mathbf{S} \subset \mathbf{B}$ .

An important analytic tool in the study of functions of class  $\mathbf{B}$  is the following

**Theorem 9.** (Eremenko–Lyubich) *For every  $f \in \mathbf{B}$ , there exists  $R > 0$  such that whenever  $|f(z)| > R$ , we have*

$$\left| z \frac{f'(z)}{f(z)} \right| \geq \frac{1}{4\pi} (\log |f(z)| - \log R).$$

Recently, Sixsmith found that this property actually characterizes class  $\mathbf{B}$ :

**Theorem 10.** *Let  $f$  be a transcendental entire function. Then either  $f \in \mathbf{B}$  and*

$$\eta := \lim_{R \rightarrow \infty} \inf_{|f(z)| > R} \left| z \frac{f'(z)}{f(z)} \right| = +\infty,$$

*or  $f \notin \mathbf{B}$  and  $\eta = 0$ .*

The proof of Theorem 10 uses Theorem 5.

Individual functions of class  $\mathbf{S}$  were studied from the point of view of the general theory of meromorphic functions by Nevanlinna, Teichmüller and others. It was found in [10] that classes  $\mathbf{S}$  and  $\mathbf{B}$  are interesting from the point of view of dynamics.

In general, entire functions, can have all sorts of dynamical pathology: they can have wandering domains, measurable invariant line fields on the Julia set, and invariant components of the set of normality which do not contain singular values, and where the iterates converge to infinity (Baker domains).

The proof of Sullivan’s non-wandering theorem depends on the fact that rational functions topologically conjugate to a given function form a manifold

of finite dimension. Thus Theorem 5 permits to extend this result to the class **S**, essentially with the same proof.

Other pathologies in dynamics of entire functions are apparently related to the complicated behavior near  $\infty$ . So the class **B** with simplest possible behavior near infinity was introduced and Theorem 6 can be used to show that for functions of class **B** the iterates cannot tend to infinity on the set of normality. So Baker domains do not exist for such functions.

These results permitted to obtain a classification of periodic components of the set of normality for functions of class **S**, similar to such classification for rational functions.

Since then, these classes were intensively studied in holomorphic dynamics. We mention only one recent result of Bishop: *functions of class B can have wandering domains*.

## 6. Further applications to dynamics

Let  $f$  be a meromorphic function, and  $z_0$  a periodic point which means that  $f^n z_0 = z_0$  for some  $n$ . The smallest  $n$  with this property is called the order of  $z_0$ , and the derivative  $\lambda = (f^n)'(z_0)$ , where  $n$  is the order, is called the multiplier. A periodic point is called attracting, repelling or neutral if  $|\lambda| < 1$ ,  $|\lambda| > 1$  or  $|\lambda| = 1$ , respectively. If  $\lambda^m = 1$  for some  $m$ , the periodic point is called neutral rational.

If  $z_0$  is a periodic point of order  $n$  then  $z_0$  is a fixed point of  $f^n$  with the same multiplier. Attracting fixed points and neutral fixed points with multiplier 1 have non-empty immediate basins of attraction. An immediate basin of attraction is defined as a maximal invariant region where  $f^m(z) \rightarrow z_0$  as  $n \rightarrow \infty$ . The following theorem of Fatou is fundamental in holomorphic dynamics:

**Theorem 11.** *Let  $D$  be an immediate basin of attraction of an attracting or of a neutral point with multiplier 1. Then the restriction  $f : D \rightarrow D$  cannot be a covering map. This means that this map has either a critical point or an asymptotic curve with asymptotic value in  $D$ . Moreover, the trajectory of some singular value in  $D$  is not absorbed by  $z_0$ .*

In the simplest case that  $z_0$  is attracting, there is a one-line proof of the first part of the theorem. Suppose that  $f : D \rightarrow D$  is a covering. Then  $f$  is a local isometry with respect to the hyperbolic metric in  $D$ . But at  $z_0$ ,  $f$  strictly compresses the hyperbolic metric. This contradiction proves the

statement.

The main corollary of Theorem 11 is that for rational functions, or for functions of class  $\mathbf{S}$ , the number of attracting and neutral rational cycles is finite: it does not exceed the number of singular values.

A component  $D$  of the set of normality is called completely invariant if  $f^{-1}(D) = D$ . The boundary of such component must coincide with the Julia set. It follows that if a meromorphic function  $f$  has at least two completely invariant components, they all must be simply connected.

If  $f$  is a rational function with at least two completely invariant components  $D_j$ , then  $f : D_j \rightarrow D_j$  are ramified coverings of degree  $d = \deg f$ , so by the Riemann–Hurwitz theorem, each  $D_j$  must contain  $d - 1$  critical points (counting with multiplicity). It follows that a rational function can have at most two completely invariant components.

How many completely invariant components can an entire meromorphic function have, is not known<sup>2</sup>. It is conjectured that the answer is at most one for transcendental entire functions and at most two for meromorphic functions, and this is known in the case of meromorphic functions of class  $\mathbf{S}$ , see [3]. For meromorphic functions of class  $\mathbf{S}$  with two completely invariant components, the Julia set is a Jordan curve [6].

### Appendix. Proof of Carleman’s inequality

Let  $v(t, \phi) = u(e^{t+i\phi})$ ; this is a sub harmonic function, and we suppose for simplicity that it is continuous. Let  $D = \{z : u(z) > 0\}$ . All integrals below are over the arcs  $\{\phi : e^{t+i\phi} \in D\}$ , and we omit  $d\phi$ . We have

$$\begin{aligned} \nu(t) &:= \int v^2 = \mu^2(t), \\ \nu' &= 2 \int vv_t, \\ \nu'' &= 2 \int (v_t^2 + vv_{tt}) \geq 2 \int (v_t^2 + v_\phi^2), \end{aligned} \tag{5}$$

where we used  $u_{tt} + u_{\phi\phi} \geq 0$  and integrated by parts.

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<sup>2</sup>Baker’s proof [2] that a transcendental entire function has at most one completely invariant domain contains a gap, [13].

Wirtinger's inequality for a  $C^1$  function which equals to zero at the end-points of an interval  $I$  says

$$|I|^2 \int_I v^2 \geq \pi^2 \int_I v_\phi^2.$$

Applying this to each maximal interval where  $v > 0$  we obtain

$$\int v_\phi^2 \geq \left(\frac{\pi}{\ell}\right)^2 \int v^2,$$

where  $\ell(t) = \theta(e^t)$ . Cauchy's inequality gives

$$(\nu')^2 = 4 \left( \int v v_t \right)^2 \leq 4 \int v^2 \int v_t^2 = 4\nu \int v_t^2.$$

Combining these two inequalities with (5) we obtain

$$\nu'' \geq \frac{(\nu')^2}{2\nu} + 2 \left(\frac{\pi}{\ell}\right)^2 \nu.$$

Rewriting this for  $\mu = \sqrt{\nu}$  we obtain

$$\mu'' \geq \left(\frac{\pi}{\ell}\right)^2 \mu,$$

which is (2).

To obtain (3) we set  $\omega = \log \mu$ . Then

$$\omega' = \frac{\mu'}{\mu}, \quad \omega'' = \frac{\mu''}{\mu} - \left(\frac{\mu'}{\mu}\right)^2,$$

so

$$\omega'' + (\omega')^2 = \frac{\mu''}{\mu} \geq \left(\frac{\pi}{\ell}\right)^2.$$

Now

$$\left(\omega' + \frac{\omega''}{2\omega'}\right)^2 \geq (\omega')^2 + \omega'' \geq \left(\frac{\pi}{\ell}\right)^2.$$

Thus

$$\omega' + \frac{\omega''}{2\omega'} \geq \frac{\pi}{\ell}.$$

Now notice that

$$\omega' + \frac{\omega''}{2\omega'} = \frac{1}{2} \frac{d}{dt} \log \left( \frac{d}{dt} e^{2\omega} \right),$$

thus

$$\frac{d}{dt} \log \left( \frac{d\nu}{dt} \right) \geq \frac{2\pi}{\theta(e^t)}.$$

Returning to the variable  $r = e^t$  and integrating this twice with respect to  $t$  we obtain (3).

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