

H. Schubert, Kalkül der abzählenden Geometrie, 1879.

Problem: *Given mp subspaces of dimension p in general position in a vector space of dimension $m + p$, how many subspaces of dimension m intersect all these given subspaces non-trivially?*

Answer for complex spaces (Schubert, 1886):

$$d(m, p) = \frac{1!2! \dots (p-1)! (mp)!}{m!(m+1)! \dots (m+p-1)!}.$$

For example:

$m =$	2	3	4	5	6
$p = 2$	2	5	14	42	132
$p = 3$		42	462	6006	87516
$p = 4$			24024	1662804	140229804

Notice that $d(m, 2)$ is the m -th Catalan number.

Grassmannian $G(m, m + p)$
 = the set of m -subspaces in \mathbf{C}^{m+p}
 = the set of $m \times (m + p)$ matrices of rank m ,
 modulo equivalence:

$$A \sim B \text{ if } A = UB, \det U \neq 0.$$

Plücker embedding:

$$G(m, m + p) \rightarrow \mathbf{P}^N, \quad N = \binom{m + p}{m} - 1.$$

All $m \times m$ minors serve as homogeneous coordinates.

An m -subspace K intersects a given p -subspace Q non-trivially iff

$$\det \begin{vmatrix} Q \\ K \end{vmatrix} = 0,$$

a *linear* equation in Plücker coordinates.

$d(m, p)$ is the number of intersections of $G(m, m + p)$ with a generic subspace of codimension mp in \mathbf{P}^N .

Projections.

Geometric definition. Let S be a projective subspace in \mathbf{P}^N of dimension $N - k - 1$. A *projection* with center S is a map

$$\phi_S : \mathbf{P}^N \setminus S \rightarrow \mathbf{P}^k. \quad (1)$$

For every point $x \in \mathbf{P}^N \setminus S$ there is a unique subspace L of dimension $N - k$ containing x and S . This L intersects a fixed projective subspace \mathbf{P}^k at a point $\phi_S(x)$.

Algebraic definition. Let A be a $(k + 1) \times (N + 1)$ matrix of maximal rank. It defines a projection (1) by

$$\phi_A(x) = Ax,$$

where x is a column of homogeneous coordinates, and S the nullspace of A .

Degree of a map $f : X \rightarrow Y$.

a) If X is oriented,

$$\deg f = \pm \sum_{x \in f^{-1}(y)} \text{sign det } f'(x).$$

b) If X is not oriented, let \tilde{X} and \tilde{Y} be the *spaces of orientations*. They are canonically oriented and

$$\tilde{X} \rightarrow X, \quad \tilde{Y} \rightarrow Y$$

are coverings with the group ± 1 . If there exists a lifting

$$\tilde{f} : \tilde{X} \rightarrow \tilde{Y},$$

which commutes with the ± 1 action, we define $\deg f = \deg \tilde{f}$.

f is called *orientable* if \tilde{f} exists.

For connected X and Y , $\deg f$ is defined as an integer, up to sign.

Theorem 1 *When $m + p$ is odd, the projections $G_{\mathbf{R}}(m, m + p) \rightarrow \mathbf{RP}^{mp}$ are orientable, with degrees independent of the centers.*

Proof of orientability. Let $G_{\mathbf{R}}^+(m, m + p)$ be the *upper Grassmannian*, that is the set of all *oriented* m -subspaces in \mathbf{R}^{m+p} . Then the coverings

$$\tilde{G}_{\mathbf{R}} \rightarrow G_{\mathbf{R}}, \quad G_{\mathbf{R}}^+ \rightarrow G_{\mathbf{R}}$$

are isomorphic. Then it is easy to check that projections lift to

$$G_{\mathbf{R}}^+ \rightarrow (\mathbf{RP}^{mp})^+,$$

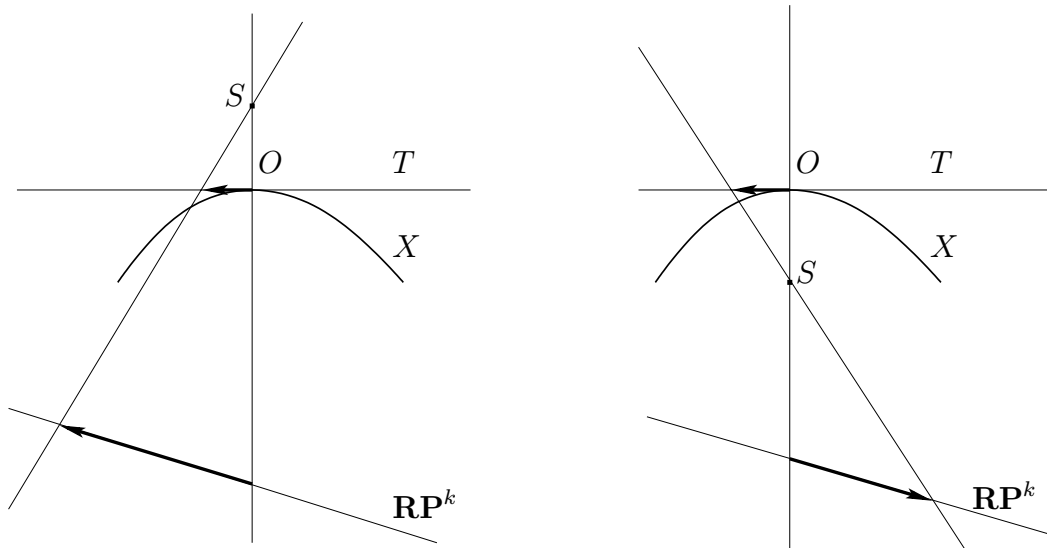
where

$$(\mathbf{RP}^{mp})^+ = \widetilde{\mathbf{RP}}^{mp}$$

is the sphere of dimension mp .

Proof of independence of the center. It works for any smooth projective variety X of even dimension, provided that projections are orientable.

For even $\dim X = k$, the degree of a projection is independent of the center:



As the center S crosses X at the point O , the images of vectors of $T_O(X)$ under $\phi'_S(O)$ change direction. As the dimension of the tangent spaces $T(X)$ and $T(\mathbf{RP}^k)$ is even, the Jacobian does not change sign.

Theorem 2 *When mp is even,
 $\deg G_{\mathbf{R}}(m, m + p) = \pm I(m, p)$, where
 $I(m, p) \neq 0$ iff $m + p$ is odd.*

Explicit expression for odd $m + p$:

$$\frac{1!2! \cdots (p-1)!(pm/2)!}{(m-p+2)!(m-p+4)! \cdots (m+p-2)!} \\ \times \frac{(m-1)!(m-2)! \cdots (m-p+1)!}{\left(\frac{m-p+1}{2}\right)! \left(\frac{m-p+3}{2}\right)! \cdots \left(\frac{m+p-1}{2}\right)!},$$

Some values of $I(m, p)$:

$m =$	3	4	5	6	7	8	9	10	11
$p = 2$	1	0	2	0	5	0	14	0	42
$p = 3$	0	2	0	12	0	110	0	1274	0
$p = 4$		0	12	0	286	0	12376	0	759696
$p = 5$			0	286	0	33592	0	8320480	0
$p = 6$				0	33592	0	...	0	...

Corollary 1 *For odd $m+p$, the intersection of $G_{\mathbf{R}}(m, m+p)$ with any projective subspace $L \subset \mathbf{RP}^N$ of codimension mp is non-empty. For a generic subspace L , this intersection contains at least $I(m, p)$ points.*

If $p = 2$, for each even m , the intersection with some subspaces L can be empty.

Corollary 2 *The number of real solutions of Schubert's problem is at least $I(m, p)$. In particular, when $m+p$ is odd, there are real solutions.*

If $p = 2$, for each even m , there are configurations of p -subspaces such that Schubert's problem has no real solutions.

Ballot sequences. For any initial segment, a has at least as many votes as b , b has at least as many as c , etc., but the election ends with a tie between all candidates.

For example,

$$\sigma = (a \ a \ b \ c \ b \ a \ b \ c \ a \ c \ b \ c).$$

The number of ballot sequences with p candidates and mp voters equals $d(m, p)$, the degree of the complex Grassman variety (Frobenius, MacMahon).

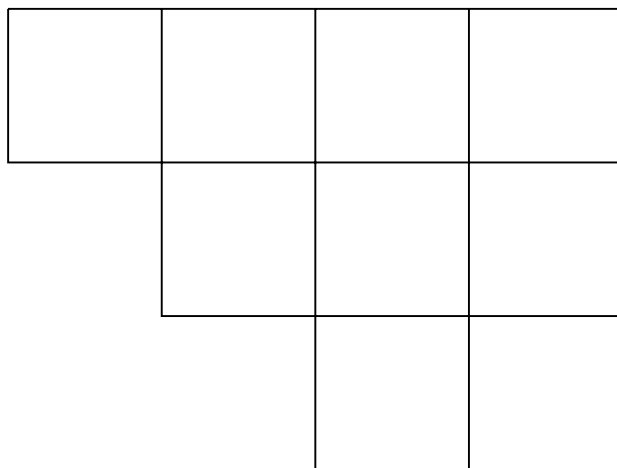
An *inversion* is a pair of ballots where the order of candidates is non-alphabetical. For the sequence above, the total number of inversions $\text{inv } \sigma = 8$.

Definition:

$$I(m, p) = \sum_{\text{all ballot seq}} (-1)^{\text{inv } \sigma}.$$

Theorem (Dennis White, 2000).

$I(m, p) = 0$ if $m + p$ is even, and for odd $m + p$, $I(m, p) =$ the number of shifted standard Young tableaux with $(m + p - 1)/2$ cells in the top row, $(m - p + 1)/2$ cells in the bottom row, and of height p :



Explicit expression for the number of such tableaux was given by Thrall in 1952.

The Wronski map $G(m, m + p) \rightarrow \mathbf{P}^{mp}$,

$$(q_1, \dots, q_p) \mapsto W[q_1, \dots, q_p].$$

Consider the $p \times (m + p)$ matrix $F(z)$ whose first row is $(z^{m+p-1}, \dots, z, 1)$ and the i -th row is the $(i - 1)$ -st derivative of the first. Then

$$W(q_1, \dots, q_p) = \begin{vmatrix} F(z) \\ K & I \end{vmatrix},$$

where

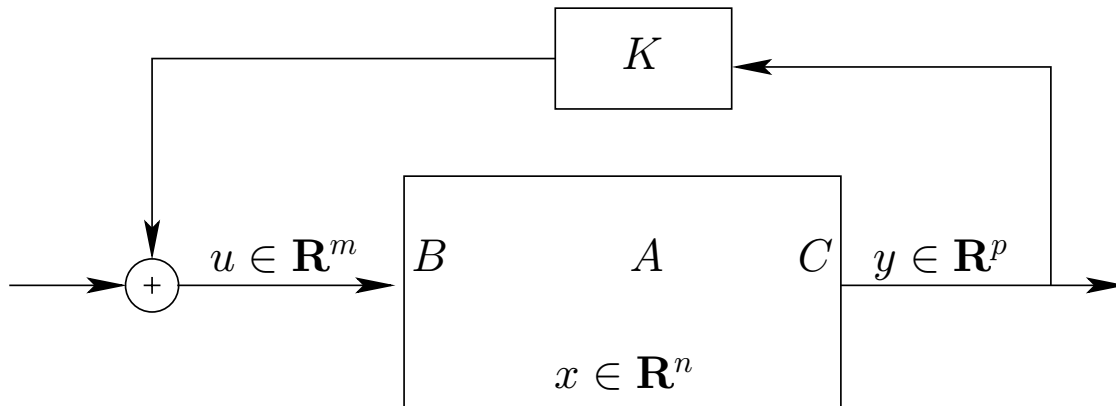
$$q_i(z) = z^{m+p-i} - \sum_{j=1}^m k_{j,i} z^{m-1},$$

so the Wronski map is *linear* in Plücker coordinates.

Theorem 3 *The degree of the Wronski map is $I(m, p)$.*

Remark. This degree is defined for all m, p .

Control of a linear system by static output feedback.



$$\begin{aligned}\dot{x} &= Ax + Bu, \\ y &= Cx, \\ u &= Ky.\end{aligned}$$

Elimination gives

$$\dot{x} = (A + BKC)x.$$

Pole placement Problem: *given real A, B, C and a real polynomial q of degree n , find real K , so that*

$$\det(\lambda I - A - BKC) = q(\lambda).$$

Brief history of automatic control

II cent. Hero of Alexandria

1780-s Watt's governor.

1831 Airy, "On the regulator..."

1868 Maxwell, "On Governors".

1877 Routh, Vyshnegradskii.

1894 Stodola asks Hurwitz.

1895 Hurwitz answers.

1914 Liénard–Chipart criterion

1932 Nyquist uses Rouché's theorem.

1981 Brockett and Byrnes use

Schubert's work.

1992 Pole placement is generically surjective if

$n > mp$ (X. Wang)

Using:

a) A coprime factorization

$$C(zI - A)^{-1}B = D(z)^{-1}N(z),$$

$$\det D(z) = \det (zI - A),$$

b) The identity

$$\det (I + PQ) = \det (I + QP),$$

we rewrite the pole placement map as

$$\begin{aligned} \psi_K(z) &= \det (zI - A - BKC) \\ &= \det (zI - A) \det (I - (zI - A)^{-1}BKC) \\ &= \det (zI - A) \det (I - C(zI - A)^{-1}BK) \\ &= \det D(z) \det (I - D(z)^{-1}N(z)K) \\ &= \det (D(z) - N(z)K) \\ &= \begin{vmatrix} D(z) & N(z) \\ K & I \end{vmatrix}, \end{aligned}$$

linear wrt Plücker coordinates.

Corollary 3 *If $m + p$ is odd, and $n = mp$, the pole placement map is surjective for a generic system with m inputs, p outputs and state of dimension n .*

Example 1 *If m is even, the pole placement map is not surjective for an open set of systems with m inputs, 2 outputs and state of dimension $2m$.*

Theorem 4 *If all critical points of a rational function belong to a circle (on the Riemann sphere), it maps this circle into a circle.*

Construction of Example 1 using Theorem 4: take simple critical points at the preimage of the image of the unit circle.

B. and M. Shapiro Conjecture: *If zeros of the Wrońskian of several polynomials are all real, the polynomials can be made real by a non-degenerate linear transformation.*

Theorem 4 implies this for two polynomials.

Proof of Theorem 4.

Equivalence of rational functions: $f \sim g$ if $f = \ell \circ g$, $\deg \ell = 1$.

Theorem (L. Goldberg, 1990) *Given $2m$ points on the Riemann sphere, there exist $d(m, 2)$ classes of rational functions with these critical points.*

Theorem 5 *Given $2m$ points on a circle, there exist $d(m, 2)$ classes of rational functions with these critical points, mapping this circle into itself.*

Corollary (Theorem 4): If all critical points belong to a circle, the function maps it into a circle.

Sketch of the proof of Theorem 5.

R is the class of rational functions,
 $f(\mathbf{T}) \subset \mathbf{T}$, $\deg f = m+1$, all critical points are
simple and belong to \mathbf{T} , $f(1) = 1$, $f'(1) = 0$.

$Net\ \gamma = f^{-1}(\mathbf{T})$, modulo orientation-preserving
homeomorphisms of $\overline{\mathbf{C}}$, fixing 1, symmetric with
respect to \mathbf{T} .

Lemma. *There are $d(m, 2)$ nets.*

A *labeling* of a net: non-negative function on
the set of edges, symmetric with respect to \mathbf{T} ,
and satisfying

$$\sum_{e \in \partial G} p(e) = 2\pi \quad \text{for every face } G.$$

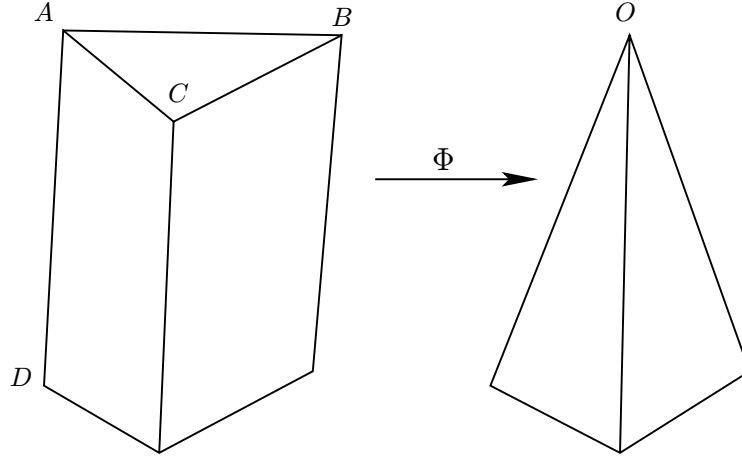
For example: $p(e) = \text{length } f(e)$.

A *critical set*: $2m$ points on \mathbf{T} , including
 $1 \in \mathbf{T}$.

Fix a net γ . Let L_γ be the space of all label-
ings and Σ_γ the space of all critical sets. They
are convex polytopes of the same dimension.
The Uniformization Theorem gives

$$\Phi_\gamma : L_\gamma \rightarrow \Sigma_\gamma.$$

This map is continuous for each γ .



Proof of surjectivity of Φ .

- a) Extension to $\overline{L}_\gamma \rightarrow \overline{\Sigma}_\gamma$.
- b) Combinatorics of the boundary map (degeneracy of rational functions)
- c) Topological lemma: *Let Φ be a continuous map of closed convex polytopes of the same dimension. If the preimage of every closed face (of any dimension) has homology groups of one point (in particular, this preimage is non-empty), then Φ is surjective.*