Moduli spaces for Lamé functions

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in collaboration with

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Results

Elliptic curve in the form of Weierstrass:

$$u^2 = 4x^3 - g_2 x - g_3, \quad g_2^3 - 27g_3^2 \neq 0$$

Lamé equation of degree m with parameters (λ, g_2, g_3):

$$\left(\left(u\frac{d}{dx}\right)^2 - m(m+1)x - \lambda\right) w = 0$$

Changing x to $x/k,\ k \in \mathbf{C}^*$ we obtain a Lamé equation with parameters

$$(k\lambda, k^2 g_2, k^3 g_3), \quad k \in \mathbf{C}^*$$

Such equations are called *equivalent*, and the set of equivalence classes is the *moduli space for Lamé equations*. It is a weighted projective space $\mathbf{P}(1,2,3)$ from which the curve $g_2^3 - 27g_3^2 = 0$ is deleted.

Elliptic form of Lamé equation

$$W'' - (m(m+1)\wp + \lambda)\, W = 0$$

is obtained by the change of the independent variable
$x = \wp(z),\ u = \wp'(z)$, so $W(z) = w(\wp(z))$.
Here $\wp$ is the Weierstrass function of the lattice Λ with invariants

$$g_2 = \sum_{\omega \in \Lambda \setminus \{0\}} \omega^{-4}, \quad g_3 = \sum_{\omega \in \Lambda \setminus \{0\}} \omega^{-6}.$$

Lamé function is a non-trivial solution w such that w^2 is a polynomial. If a Lamé function exists, it is unique up to a constant factor. It exists iff a polynomial equation holds

$$F_m(\lambda, g_2, g_3) = 0$$

This polynomial is quasi-homogeneous with weights $(1, 2, 3)$ so we can factor by the $\mathbf{C}^*$ action $(\lambda, g_2, g_3) \mapsto (k\lambda, k^2 g_2, k^3 g_3)$, and obtain an (abstract) Riemann surface $\mathbf{L}_m$, the *moduli space of Lamé functions*. The map

$$(g_2, g_3) \mapsto J = g_2^3/(g_2^3 - 27g_3^2)$$

is homogeneous, so it defines a function

$$\pi : \mathbf{L}_m \to \mathbf{C}_J$$

which is called the *forgetful map*.

Singularities in $\mathbf{C}$ of Lamé's equation in algebraic form are e_1, e_2, e_3,

$$4x^2 - g_2 x - g_3 = 4(x - e_1)(x - e_2)(x - e_3)$$

with local exponents $(0, 1/2)$. So Lamé functions are of the form:

$$Q(x), \quad Q(x)\sqrt{(x - e_i)(x - e_j)}, \quad m \text{ even},$$

or

$$Q(x)\sqrt{x - e_i}, \quad Q(x)\sqrt{(x - e_1)(x - e_2)(x - e_3)}, \quad m \text{ odd},$$

where Q is a polynomial. This shows that for every $m \geq 2$, $\mathbf{L}_m$ consists of *at least two* components.

We determine topology of $\mathbf{L}_m$ (number of connected components, their genera and numbers of punctures).

Notation:

$$d_m^I := \begin{cases} m/2+1, & m \text{ even} \\ (m-1)/2, & m \text{ odd} \end{cases}$$

$$d_m^{II} := 3\lceil m/2 \rceil.$$

$$\epsilon_0 := 0, \text{ if } m \equiv 1 \pmod 3, \text{ and 1 otherwise}$$

$$\epsilon_1 := 0, \text{ if } m \in \{1,2\} \pmod 4, \text{ and 1 otherwise}$$

Orbifold Euler characteristic is defined as

$$\chi^O = 2 - 2g - \sum_z \left(1 - \frac{1}{n(z)}\right),$$

where g is the genus, and $n : S \to \mathbf{N}$ the orbifold function.

Theorem 1. *For $m \geq 2$, $\mathbf{L}_m$ has two components, $\mathbf{L}_m^I$ and $\mathbf{L}_m^{II}$. They have a natural orbifold structure with ϵ_0 points of order 3 in $\mathbf{L}_m^I$ and one point of order 2 which belongs to $\mathbf{L}^I$ when $\epsilon_1 = 1$ and to $\mathbf{L}_m^{II}$ otherwise.*
Component I has d_m^I punctures and component II has $2d_m^{II}/3 = 2\lceil m/2 \rceil$ punctures.
The degrees of forgetful maps are d_m^I and d_m^{II}.
The orbifold Euler characteristics are

$$\chi^O(\mathbf{L}_m^I) = -(d_m^I)^2/6, \quad \chi^O(\mathbf{L}_m^{II}) = -(d_m^{II})^2/18$$

For $m = 0$ there is only the first component and for $m = 1$ only the second component. So $\mathbf{L}_m$ is connected for $m \in \{0, 1\}$.

That there are at least two components is well-known. The new result is that there are exactly two, and their Euler characteristics.

Corollary 1. *The polynomial F_m factors into two irreducible factors in* $\mathbf{C}(\lambda, g_2, g_3)$

Theorem 2. *All singular points of irreducible components of the surface $F_m = 0$ are contained in the lines* $(0, t, 0)$ *and* $(0, 0, t)$.

To prove this, we find non-singular curves $\overline{\mathbf{H}}^j_m \subset \mathbf{P}^2$ and orbifold coverings $\Psi^K_m : \overline{\mathbf{H}}^j_m \to \overline{\mathbf{L}}^K_m$. Here $\overline{\mathbf{L}}^K_m$ is the compactification obtained by filling the punctures and assigning an appropriate orbifold structure at the punctures. Theorem 1 is used to prove non-singularity of $\mathbf{H}^j_m$.

We thank Vitaly Tarasov (IUPUI) and Eduardo Chavez Heredia (Univ. of Bristol) who helped us to find ramification of π over $J = 0$. Tarasov also suggested the definition of $\overline{\mathbf{H}}^j_m$ which is crucial here.

Let $F_m = F_m^I F_m^{II}$ and let $D^I,\ D^{II}$ be discriminants of $F_m^I,\ F_m^{II}$ with respect to λ. These are quasi-homogeneous polynomials, so equations $D_m^K = 0$ are equivalent to polynomial equations $C_m^K(J) = 0$ in one variable. These C_m^K are called *Cohn's polynomials*.

Corollary 2. (conjectured by Robert Meier)
deg $C_m^I = \lfloor (d_m^2 - d_m + 4)/6 \rfloor,\ d = d_m^I$ and
deg $C_m^{II} = d_m^{II}(d_m^{II} - 1)/2.$

Since we know the genus of $\mathbf{L}_m^K$ (Theorem 1), we can find ramification of the forgetful map $\pi : \mathbf{L}_m \to \mathbf{C}_J$. Degree of C_m^K differs from this ramification by contribution from singular points of $F_m^K = 0$, and this contribution is obtained from Theorem 2.

Method

Let w be a Lamé function. Then a linearly independent solution of the same equation is $w \int dx/(uw^2)$, so their ratio

$$f = \int w^{-2}(x) \frac{dx}{u}$$

is an Abelian integral. The differential df has a single zero of order $2m$ and m double poles with *vanishing residues*.
Conversely, if $g(x)dx$ is an Abelian differential on an elliptic curve with a single zero at the origin[1] of multiplicity $2m$ and m simple poles with vanishing residues, then $g = 1/(uw^2)$ where w is a Lamé function. Such differentials on elliptic curves are called *translation structures*. They are defined up to proportionality.

[1] The "origin" is a neutral point of the elliptic curve. It corresponds to $x = \infty$ in Weierstrass representation

So we have a $1-1$ correspondence between Lamé functions and translation structures.

To study translation structures we pull back the *Euclidean* metric from $\mathbf{C}$ to our elliptic curve via f, so that f is the *developing map* of the resulting metric. This metric is flat, has one conic singularity with angle $2\pi(2m+1)$ at the origin, and m simple poles.

A *pole* of a flat metric is a point whose neighborhood is isometric to $\{z : R < |z| \leq \infty\}$ with flat metric, for some $R > 0$.

We have a $1-1$ correspondence between the classes of Lamé functions and the classes of such metrics on elliptic curves. (Equivalence relation of the metrics is proportionality. In terms of the developing map, $f_1 \sim f_2$ if $f_1 = Af_2 + B,\ A \neq 0$.)

We will show that every *flat singular torus* (a torus with a metric described above) can be cut into two *congruent* flat singular triangles in an essentially unique way.

A *flat singular triangle* is a triple $(\Delta, \{a_j\}, f)$, where D is a closed disk, a_j are three (distinct) boundary points, and f is a meromorphic function $\Delta \to \overline{\mathbf{C}}$ which is locally univalent at all points of Δ except a_j, has *conic singularities* at a_j,

$$f(z) = f(a_j) + (z - a_j)^{\alpha_j} h_j(z), \quad h_j \text{ analytic}, \quad h_j(a_j) \neq 0$$

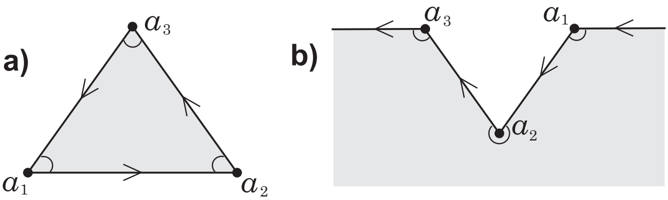
$f(a_j) \neq \infty$, and the three arcs (a_i, a_{i+1}) of $\partial\Delta$ are mapped into lines ℓ_j (which may coincide).
The number $\pi\alpha_j > 0$ is called the *angle* at the *corner* a_j.

Flat singular triangles $(\Delta_1, \{a_j\}, f_1)$ and $(\Delta_2, \{a'_j\}, f_2)$ are *equivalent* if there is a conformal homeomorphism $\phi : \Delta_1 \to \Delta_2, \ \phi(a_j) = a'_j$, and

$$f_2 = Af_1 \circ \phi + B, \quad A \neq 0.$$

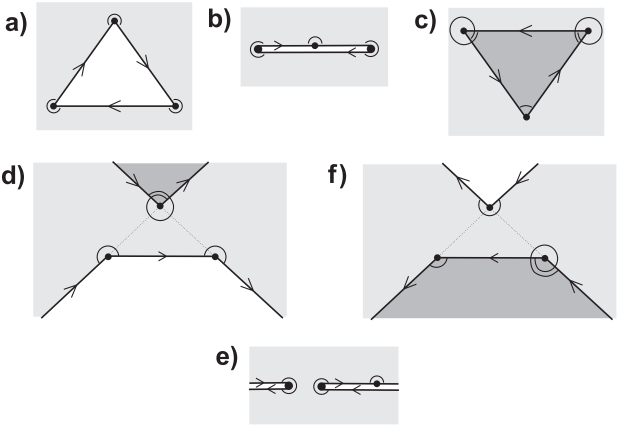
To visualize, draw three lines ℓ_j in the plane, not necessarily distinct, choose three distinct points $a_i \in \ell_j \cap \ell_k$, and mark the angles at these points with little arcs (the angles are positive but can be arbitrarily large).

The corners a_j are enumerated according to the positive orientation of $\partial\Delta$.



"Primitive" triangles with angle sums π and 3π

All other balanced triangles can be obtained from these two by gluing half-planes to the sides (F. Klein).



All types of balanced triangles with angle sum 5π ($m = 2$)

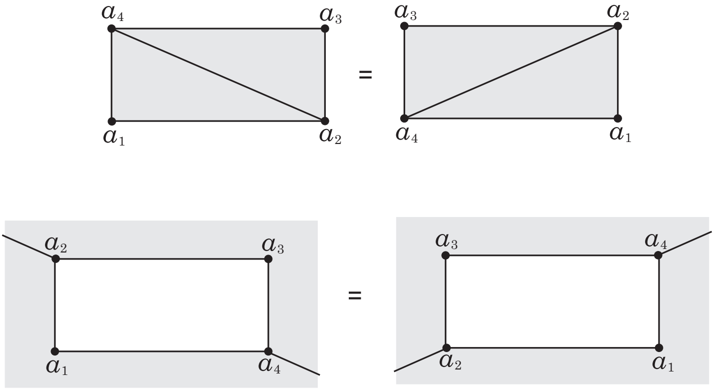
A flat singular triangle is called *balanced* if

$$\alpha_i \leq \alpha_j + \alpha_k$$

for all permutations (i, j, k), and *marginal* if we have equality for some permutation. We abbreviate them as BFT.

Let T be a BFT and T' its congruent copy. We glue them by identifying the pairs of equal sides according to the *orientation-reversing isometry.* The resulting torus is called $\Phi(T)$. All three corners of T are glued into one point, the conic singularity of $\Phi(T)$.

When two different triangles give the same torus?
a) when they differ by cyclic permutation of corners a_j, or
b) they are marginal, and are reflections of each other.



Non-uniqueness of decomposition of a torus into marginal triangles for $m = 0$ and $m = 1$ (Case b). For triangles with the angle sum π or 3π, marginal means that the largest angle is $\pi/2$ or $3\pi/2$.

Complex analytic structure on the space $\mathbf{T}_m$ of BFT:
A complex local coordinate is the ratio

$$z = \frac{f(a_i) - f(a_j)}{f(a_k) - f(a_j)}$$

There are 6 such coordinates and they are related by transformations of the anharmonic group:

$$z, \ 1/z, \ 1 - z, \ 1 - 1/z, \ 1/(1 - z), \ z/(z - 1)$$

This coordinate z is also the ratio of the periods of the Abelian differential $dx/(uw^2)$ corresponding to a Lamé function.

Factoring the space $\mathbf{T}_m$ of BFT's with the angle sum $\pi(2m+1)$ by equivalences a) and b) we obtain the space $\mathbf{T}_m^*$. It inherits the complex analytic structure from $\mathbf{T}_m$. Our main result is

Theorem 3. $\Phi : \mathbf{T}_m^* \to \mathbf{L}_m$ *is a conformal homeomorphism.*

Roughly speaking, every flat singular torus can be broken into two congruent BFT, and this decomposition is unique modulo equivalences a) and b).

The space $\mathbf{T}_m^*$ has a nice partition into open 2- and 1- cells and points, which permits to compute the topological characteristics of $\mathbf{L}_m$. To explain this partition, we study BFT.

Some properties of BFT.
1. The sum of the angles is an odd multiple of π. The angles are $\pi\alpha_j$ where either all α_j are integers or none of them is an integer.
2. For non-integer α_j, triangle is determined by the angles, and any triple of positive non-integer α_j whose sum is odd can occur.
3. For integer angles, all triangles are balanced. Triangle is determined by the angles and one real parameter (for example the ratio z introduced above). All balanced integer triples whose sum is an odd can occur as α_j.
4. For BFT, each side contains at most one pole, and

$$2n + k = m,$$

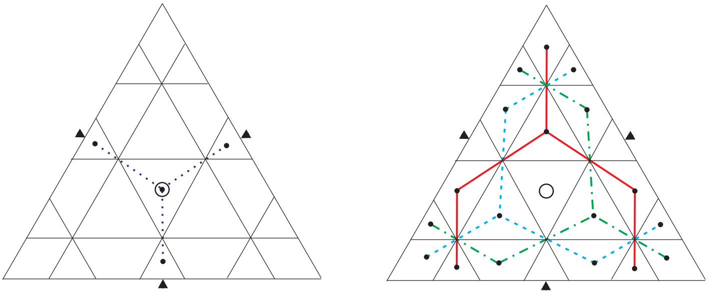
where n is the number of interior poles, k is the number of poles on the sides, and $\pi(2m + 1)$ is the sum of the angles.

The two components are determined by the value of k: for example, when m is even then $k = 0$ on the first component and $k = 2$ on the second.

To visualize 2 and 3, consider the *space of angles* $\mathbf{A}_m$. First we define the triangle

$$\Delta_m = \{\alpha \in \mathbf{R}^3 : \sum_1^3 \alpha_j = 2m + 1,\ 0 < \alpha_i \leq \alpha_j + \alpha_k\},$$

then remove from it all lines $\alpha_j = k$, for integer k, and then add all integer points (where all α_j are integers). The resulting set is the space of angles $\mathbf{A}_m$.



Space of angles $\mathbf{A}_3$ and 4 components of $\mathbf{T}_3$ (Their "nerves" are shown). The three components on the right-hand side are identified when we pass to the factor $\mathbf{T}^*$.

We have a map $\psi : \mathbf{T}_m \to \mathbf{A}_m$ which to every triangle puts into correspondence its vector of angles (divided by π). This map is $1-1$ on the set of triangles with angles non-integer multiples of π. Preimage of an integer point in $\mathbf{A}_m$ consists of three open intervals.

This defines a natural partition of $\mathbf{T}_m$ into open disks and intervals. Open disks are preimages of components of interior of $\mathbf{A}_m$, intervals are of two types: inner edges are components of preimages of integer points in $\mathbf{A}_n$, and boundary edges are preimages of the intervals $\mathbf{A}_m \cap \partial\Delta_m$.

This partition reduces calculation of Euler characteristic and number of punctures to combinatorics.

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