Moduli spaces for Lamé functions

Speaker: A. Eremenko (Purdue University)

in collaboration with

Andrei Gabrielov (Purdue, West Lafayette, IN),
Gabriele Mondello ("Sapienza", Università di Roma)
and Dmitri Panov (King's College, London)

Thanks to: Walter Bergweiler, Vitaly Tarasov and Eduardo Chavez Heredia.

Seminar "Geometric Analysis and Differential Geometray",
Princeton, February 2021

Advertisement

1. Linear ODE, special functions, planar non-singular curves. A conjecture of Robert Maier.

2. Abelian differentials of the second kind with one zero and vanishing residues. Translation structures on tori.

3. Flat triangles.

4. Application (and original motivation): spherical metrics with one conic singularity on tori: their degeneration.

Introduction

Lamé equation is a second order linear ODE with 4 regular singularities, and the Riemann scheme

$$\left(\begin{array}{ccccc} e_1 & e_2 & e_3 & \infty & \\ 0 & 0 & 0 & -m/2; & x \\ 1/2 & 1/2 & 1/2 & (m+1)/2 & \end{array} \right),$$

where $m \geq 0$ is an integer. By a change of the independent variable it can be rewritten as an equation on a torus with single singularity with trivial local monodromy. It has one accessory parameter which makes it the simplest Fuchsian equation, after the hypergeometric one.

The preceding five chapters have been occupied with the discussion of functions which belong to what may be generally described as the hypergeometric type, and many simple properties of these functions are now well known.
In the present chapter we enter upon a region of Analysis which lies beyond this, and which is, as yet, only very imperfectly explored.

(E. T. Whittaker and G. N. Watson, A course of modern analysis, 1927 edition, introduction to Ch. XIX.)

It was discovered by Gabriel Lamé in 1839[1] when separating variables in the Laplace equation in the ellipsoidal coordinates in R^3. Lam'e functions plays the same role for ellipsoidal coordinates as Legendre's functions for spherical coordinates. Lamé equation was studied much in 19th and 20th centuries, mainly for the case when all parameters are real.

[1] For historical perspective, 1838: coronation of Queen Victoria, 1840: abolition of slave srade in UK

Gabriel Lamé (1795–1870)

Elliptic curve form of the Lamé equation

Elliptic curve in the form of Weierstrass:

$$u^2 = 4x^3 - g_2 x - g_3 = 4(x - e_1)(x - e_2)(x - e_3),$$

$$g_2^3 - 27 g_3^2 \neq 0.$$

Lamé equation of degree m with parameters (λ, g_2, g_3) on this elliptic curve is

$$\left(\left(u \frac{d}{dx} \right)^2 - m(m+1)x - \lambda \right) w = 0$$

This is a linear ODE on a torus with respect to a function w, with one regular singularity at ∞. λ is called the *accessory parameter*.

Two other forms

To obtain a Lamé equation on the sphere, we just open parentheses in $(ud/dx)^2$ and insert the expression of u:

$$w'' + \frac{1}{2}\left(\sum_{k=1}^{3} \frac{1}{x - e_j}\right) w' = \frac{m(m+1)x + \lambda}{4(x-e_1)(x-e_2)(x-e_3)} w.$$

This corresponds to the Riemann scheme written in the beginning.
Elliptic functions form of the Lamé equation
is obtained by the change of the independent variables in the previous forms: $x = \wp(z)$, $u = \wp'(z)$, so $W(z) = w(\wp(z))$.

$$W'' - (m(m+1)\wp + \lambda)\, W = 0$$

Here $\wp$ is the Weierstrass function of the lattice Λ with invariants

$$g_2 = 60 \sum_{\omega \in \Lambda \setminus \{0\}} \omega^{-4}, \quad g_3 = 140 \sum_{\omega \in \Lambda \setminus \{0\}} \omega^{-6}.$$

Changing x to x/k, $k \in \mathrm{C}^*$ we obtain a Lamé equation with parameters

$$(k\lambda, k^2 g_2, k^3 g_3), \quad k \in \mathrm{C}^*$$

Such equations are called *equivalent*, and the set of equivalence classes is the *moduli space for Lamé equations* Lame_m.
It is a weighted projective space $\mathrm{P}(1,2,3)$ from which the curve $g_2^3 - 27g_3^2 = 0$ is deleted. Since the function

$$J = \frac{g_2^3}{g_2^3 - 27g_3^2}$$

is homogeneous, it defines a map $\pi_m : \mathrm{Lame}_m \to \mathrm{C}_J$ which is called the *forgetful map*. The complex plane C_J is the moduli space of elliptic curves.

A Lamé function

is a non-trivial solution w such that w^2 is a polynomial (or $W^2 = w^2 \circ \wp$ is an even elliptic function). If a Lamé function exists, it is unique up to a constant factor. It exists iff a polynomial equation holds

$$F_m(\lambda, g_2, g_3) = 0$$

This polynomial is monic in λ and quasi-homogeneous with weights $(1, 2, 3)$ so we can factor by the C^* action

$$(\lambda, g_2, g_3) \mapsto (k\lambda, k^2 g_2, k^3 g_3),$$

and obtain a curve in Lame_m whose normalization is an (abstract) Riemann surface L_m, the *moduli space of Lamé functions*.

Singularities in C of Lamé's equation in algebraic form are e_1, e_2, e_3,

$$4x^2 - g_2 x - g_3 = 4(x - e_1)(x - e_2)(x - e_3)$$

with local exponents $(0, 1/2)$. So Lamé functions are of the form:

$$Q(x), \quad Q(x)\sqrt{(x - e_i)(x - e_j)}, \quad m \text{ even},$$

or

$$Q(x)\sqrt{x - e_i}, \quad Q(x)\sqrt{(x - e_1)(x - e_2)(x - e_3)}, \quad m \text{ odd},$$

where Q is a polynomial. This shows that for every $m \geq 2$, L_m consists of *at least two* components. We call L_m^I the part which is invariant under permutations of e_1, e_2, e_3 and the rest is L_m^{II}.

We determine topology of L_m (number of connected components, their genera and numbers of punctures).

The language of orbifolds will be convenient.
An *orbifold* is a compact Riemann surface S equipped with a function $n : S \to \mathrm{N} \cup \{\infty\}$ which equals 1 at all points except finitely many.
Orbifold Euler characteristic is

$$\chi^O = 2 - 2g - \sum_z \left(1 - \frac{1}{n(z)}\right),$$

where g is the genus.
A ramified covering $\psi : S_1 \to S_2$ is called an *orbifold map* if $n_2(f(z))$ divides $n_1(z)\deg_z f$, and it is called an *orbifold covering* if

$$n_2(f(z)) = n_1(z)\deg_z f \quad \text{for all} \quad z \in S_1.$$

Notation:

$$d_m^I := \begin{cases} m/2+1, & m \text{ even} \\ (m-1)/2, & m \text{ odd} \end{cases}$$

$$d_m^{II} := 3\lceil m/2 \rceil.$$

These are the degrees of the forgetful maps.

$$\epsilon_0 := 0, \text{ if } m \equiv 1 \pmod 3, \text{ and 1 otherwise}$$

$$\epsilon_1 := 0, \text{ if } m \in \{1,2\} \pmod 4, \text{ and 1 otherwise}$$

In other words, $\epsilon_0 = 0$ iff d_m^I is divisible by 3, and $\epsilon_1 = 0$ iff d_m^I is even.

Theorem 1. *For* $m \geq 2$, L_m *has two components,* L_m^I *and* L_m^{II}*. They have a natural orbifold structure with* ϵ_0 *points of order* 3 *in* L_m^I*, and one point of order* 2 *which belongs to* L^I *when* $\epsilon_1 = 1$ *and to* L_m^{II} *otherwise.*
Component I has d_m^I *punctures and component II has* $2d_m^{II}/3 = 2\lceil m/2 \rceil$ *punctures.*
The degrees of forgetful maps are d_m^I *and* d_m^{II}*.*
The orbifold Euler characteristics are

$$\chi^O(\mathsf{L}_m^I) = -(d_m^I)^2/6, \quad \chi^O(\mathsf{L}_m^{II}) = -(d_m^{II})^2/18.$$

For $m = 0$ *there is only the first component and for* $m = 1$ *only the second component. So* L_m *is connected for* $m \in \{0, 1\}$*.*

That there are at least two components is well-known. The new result is that there are exactly two, and their Euler characteristics.

Corollary 1. *The polynomial* F_m *factors into two irreducible factors in* $\mathsf{C}(\lambda, g_2, g_3)$

Theorem 2. *All singular points of irreducible components of the surface* $F_m = 0$ *are contained in the lines* $(0, t, 0)$ *and* $(0, 0, t)$.

To prove this, we find non-singular curves $\overline{\mathsf{H}}^j_m \subset \mathsf{P}^2$ and orbifold coverings $\Psi^K_m : \overline{\mathsf{H}}^j_m \to \overline{\mathsf{L}}^K_m$. Here $\overline{\mathsf{L}}^K_m$ is the compactification obtained by filling the punctures and assigning an appropriate orbifold structure at the punctures. Theorem 1 is used to prove non-singularity of H^j_m.

We thank Vitaly Tarasov (IUPUI) and Eduardo Chavez Heredia (Univ. of Bristol) who helped us to find ramification of π *over* $J = 0$. *Tarasov also suggested the definition of* $\overline{\mathsf{H}}^j_m$ *which is crucial here.*

Let $F_m = F_m^I F_m^{II}$ and let $D^I,\ D^{II}$ be discriminants of $F_m^I,\ F_m^{II}$ with respect to λ. These are quasi-homogeneous polynomials, so equations $D_m^K = 0$ are equivalent to polynomial equations $C_m^K(J) = 0$ in one variable. These C_m^K are called *Cohn's polynomials*.

Corollary 2. (conjectured by Robert Meier)
deg $C_m^I = \lfloor (d_m^2 - d_m + 4)/6 \rfloor,\ d = d_m^I$ and
deg $C_m^{II} = d_m^{II}(d_m^{II} - 1)/2$.

Since we know the genus of L_m^K (Theorem 1), we can find ramification of the forgetful map $\pi : \mathrm{L}_m \to \mathrm{C}_J$. Degree of C_m^K differs from this ramification by contribution from singular points of $F_m^K = 0$, and this contribution is obtained from Theorem 2.

Method

Let w be a Lamé function. Then a second linearly independent solution of the same equation is $w \int dx/(uw^2)$, so their ratio

$$f = \int \frac{dx}{u(x)w^2(x)} \quad (u^2 = 4x^3 - g_2 x - g_3)$$

is an Abelian integral. The differential df has a single zero of order $2m$ and m double poles with *vanishing residues*.
Conversely, if $g(x)dx$ is an Abelian differential on an elliptic curve with a single zero at the origin[2] of multiplicity $2m$ and m double poles with vanishing residues, then $g = 1/(uw^2)$ where w is a Lamé function. Such differentials on elliptic curves are called *translation structures*. They are defined up to proportionality.

[2]The "origin" is a neutral point of the elliptic curve. It corresponds to $x = \infty$ in Weierstrass representation

So we have a $1-1$ correspondence between Lamé functions and translation structures.

To study translation structures we pull back the *Euclidean* metric from C to our elliptic curve via f, so that f becomes the *developing map* of the resulting metric. This metric is flat, has one conic singularity with angle $2\pi(2m+1)$ at the origin, and m simple poles.

A *pole* of a flat metric is a point whose neighborhood is isometric to $\{z : R < |z| \leq \infty\} \subset \mathrm{C} \cup \{\infty\}$ with flat metric, for some $R > 0$.

We have a $1-1$ correspondence between the classes of Lamé functions and the classes of such metrics on elliptic curves. (Equivalence relation of the metrics is proportionality. In terms of the developing map, $f_1 \sim f_2$ if $f_1 = Af_2 + B,\ A \neq 0$.)

Main technical result:

Theorem. *Every flat singular torus with one conic point with angle* $(2\pi)(2m+1)$ *can be cut into two* congruent *flat singular triangles in an essentially unique way.*

"Congruent" means corresponding by an isometry *preserving orientation*.
A *flat singular triangle* is a triple $(\Delta, \{a_j\}, f)$, where D is a closed disk, a_j are three (distinct) boundary points, and f is a meromorphic function $\Delta \to \overline{\mathbf{C}}$ which is locally univalent at all points of Δ except a_j, has *conic singularities* at a_j,

$$f(z) = f(a_j) + (z - a_j)^{\alpha_j} h_j(z), \quad h_j \text{ analytic}, \quad h_j(a_j) \neq 0$$

$f(a_j) \neq \infty$, and the three arcs (a_i, a_{i+1}) of $\partial\Delta$ are mapped into lines ℓ_j (which may coincide).
The number $\pi\alpha_j > 0$ is called the *angle* at the *corner* a_j.

Flat singular triangles $(\Delta_1, \{a_j\}, f_1)$ and $(\Delta_2, \{a_j'\}, f_2)$ are *equivalent* if there is a conformal homeomorphism $\phi : \Delta_1 \to \Delta_2,\ \phi(a_j) = a_j'$, and

$$f_2 = Af_1 \circ \phi + B, \quad A \neq 0.$$

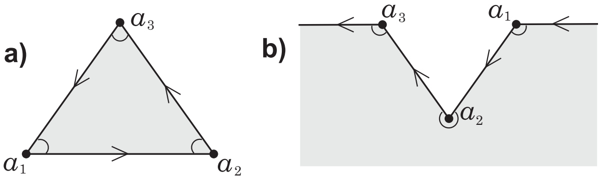
To visualize, draw three lines ℓ_j in the plane, not necessarily distinct, choose three distinct points $a_i \in \ell_j \cap \ell_k$, and mark the angles at these points with little arcs (the angles are positive and can be arbitrarily large).

The corners a_j are enumerated according to the positive orientation of $\partial\Delta$.

A flat singular triangle is called *balanced* if

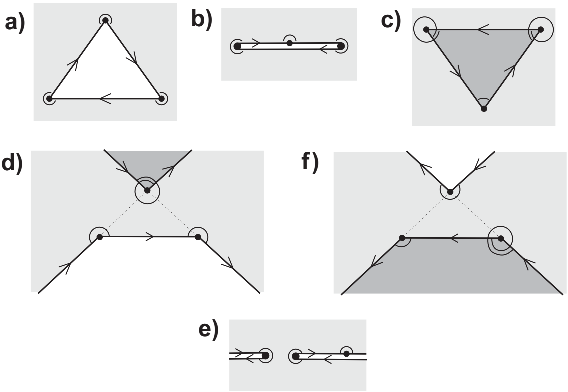
$$\alpha_i \leq \alpha_j + \alpha_k$$

for all permutations (i, j, k), and *marginal* if we have an equality.



"Primitive" triangles with angle sums π and 3π

All other balanced triangles can be obtained from these two by gluing half-planes to the sides (F. Klein).

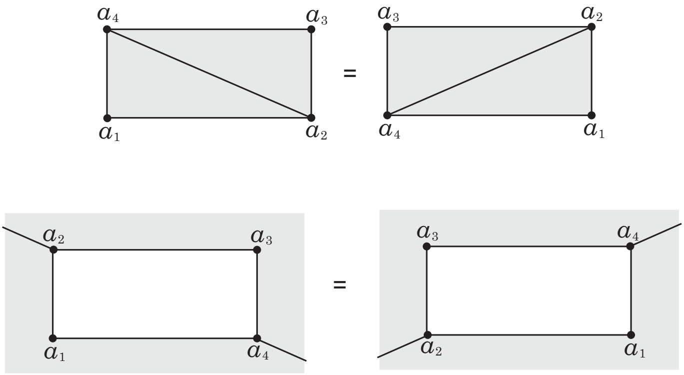


All types of balanced triangles with angle sum 5π $(m = 2)$

We abbreviate "balanced flat singular triangle" as BFT.

Let T be a BFT and T' its congruent copy. We glue them by identifying the pairs of equal sides according to the *orientation-reversing isometry.* The resulting torus is called $\Phi(T)$. All three corners of T are glued into one point, the conic singularity of $\Phi(T)$.

When two different triangles give the same torus?
a) when they differ by cyclic permutation of corners a_j, or
b) they are marginal, and are reflections of each other.



Non-uniqueness of decomposition of a torus into marginal triangles for $m = 0$ and $m = 1$ (Case b). For triangles with the angle sum π or 3π, marginal means that the largest angle is $\pi/2$ or $3\pi/2$.

Complex analytic structure on the space T_m *of BFT*

A complex local coordinate is the ratio

$$z_{i,j,k} = \frac{f(a_i) - f(a_j)}{f(a_k) - f(a_j)}$$

There are 6 such coordinates and they are related by transformations of the anharmonic group:

$$z,\ 1/z,\ 1-z,\ 1-1/z,\ 1/(1-z),\ z/(z-1)$$

Coordinates $z_{i,j,k}$ ares ratios of the periods of the Abelian differential $dx/(uw^2)$ corresponding to a Lamé function.

Factoring the space T_m of BFT's with the angle sum $\pi(2m+1)$ by equivalences a) and b) we obtain the space T_m^*. It inherits the complex analytic structure from T_m. Our main result is

Theorem 3. $\Phi : T_m^* \to L_m$ *is a conformal homeomorphism.*

Roughly speaking, every flat singular torus can be broken into two congruent BFT, and this decomposition is unique modulo equivalences a) and b).

The space T_m^* has a nice partition into open 2- and 1- cells and points, which permits to compute the topological characteristics of L_m. To explain this partition, we study BFT.

Some properties of BFT.

1. The sum of the angles is an odd multiple of π. The angles are $\pi\alpha_j$ where either all α_j are integers or none of them is an integer.

2. For non-integer α_j, triangle is determined by the angles, and any triple of positive non-integer α_j whose sum is odd can occur.

3. For integer angles, all triangles are balanced. Triangle is determined by the angles and one real parameter (for example a ratio $z_{i,j,k}$ introduced above). All balanced integer triples whose sum is odd can occur as α_j.

4. For BFT, each side contains at most one pole, and

$$2n + k = m,$$

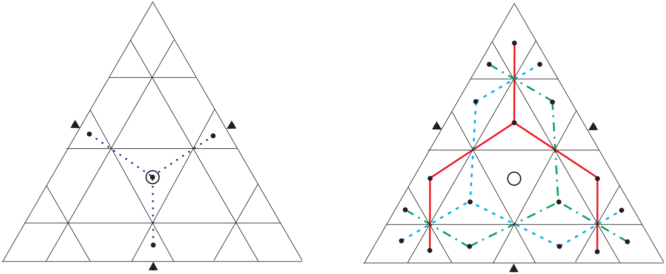
where n is the number of interior poles, k is the number of poles on the sides, and $\pi(2m + 1)$ is the sum of the angles.

The two components $\mathsf{L}_m^I, \mathsf{L}_m^{II}$ are determined by the value of k: for example, when m is even then $k = 0$ on the first component and $k = 2$ on the second.

To visualize 2 and 3, consider the *space of angles* A_m. First we define the triangle

$$\Delta_m = \{\alpha \in \mathsf{R}^3 : \sum_1^3 \alpha_j = 2m + 1,\ 0 < \alpha_i \leq \alpha_j + \alpha_k\},$$

then remove from it all lines $\alpha_j = k$, for integer k, and then add all integer points (where all α_j are integers). The resulting set is the space of angles A_m.



Space of angles A_3 and 4 components of T_3 (Their "nerves" are shown). The three components on the right-hand side are identified when we pass to the factor T^*.

We have a map $\psi : \mathrm{T}_m \to \mathrm{A}_m$ which to every triangle puts into correspondence its vector of angles (divided by π). This map is $1-1$ on the set of triangles with angles non-integer multiples of π. Preimage of an integer point in A_m consists of three open intervals.

This defines a natural partition of T_m into open disks and intervals. Open disks are preimages of components of interior of A_m, intervals are of two types: inner edges are components of preimages of integer points in A_n, and boundary edges are preimages of the intervals $\mathrm{A}_m \cap \partial\Delta_m$. They correspond to marginal triangles.

This partition reduces calculation of Euler's characteristics and numbers of punctures to combinatorics.

Proof of Maier's conjecture on degrees of Cohn's polynomials

Once we know the topology of L_m and the degree of the forgetful map $\mathrm{L}_m \to \mathrm{C}_J$, ramification is obtained by the Riemann-Hurwitz formula. Hexagonal ($J = 0$) and square ($J = 1$) tori require special investigation since our curve might be singular at those points. This investigation is based on the following

Lemma. *Let $(a_{i,j})_{i,j=1}^n$ be a matrix with $a_{j,j+1} > 0$, $a_{i+2,i} > 0$, the rest of the entries are 0. Then the characteristic polynomial has the form $\lambda^k P(\lambda^3)$, where $k \in \{0, 1, 2\}$ and P is a real polynomial with all roots negative and distinct.*

This lemma, proved by V. Tarasov, is a generalization of the classical result on Jacobi matrices.

Application to spherical metrics

Every Riemann surface has a complete Riemannian metric of constant curvature $0, 1$ *or* -1*; it is unique up to scaling in the first case, and unique in the other two cases.*
For curvature 0 or -1, this result has been extended to metrics with conic singularities by E. Picard (1889). The case of positive curvature remains wide open.

The simplest case is that of a torus S with one conic singularity with angle $2\pi\alpha,\ \alpha > 1$. Developing map $f : L \to \overline{\mathbb{C}}$ of such a metric is a ratio of two linearly independent solutions of a Lamé equation with *unitarizable* projective monodromy. So the moduli space $\mathrm{Sph}_{1,1}(\alpha)$ of such metrics can be identified with a subset of the moduli space of Lamé equations $\mathrm{P}(1, 2, 3)$.

A Riemannian metric defines a conformal structure, so we have the *forgetful map*

$$\pi : \mathrm{Sph}_{1,1}(\alpha) \to \mathsf{C}_J,$$

where C_J is the moduli space of elliptic curves.
The map π is surjective and proper, for all $\alpha > 0$, *except odd integers* (Chang-Shou Lin, G. Mondello and D. Panov), but $\mathrm{Sph}_{1,1}(2m+1)$ has a boundary in $\mathsf{P}(1,2,3)$. It is a real-analytic surface, not a complex curve, and the forgetful map restricted to $\mathrm{Sph}_{1,1}(2m+1)$ is not holomorphic.
It turns out that this boundary is a subset of L (the set of those Lamé equations which have Lamé functions as solutions) and this subset is characterized by the property that the periods of the corresponding Abelian differential have *real ratio*.
In our correspondence $\Phi : \mathsf{T}_m \to \mathsf{L}_m$ the boundary $\partial\mathrm{Sph}_{1,1}(2m+1)$ corresponds to triangles with all angles integer multiples of π.

Lin–Wang curves

The components of $\partial \mathrm{Sph}_{1,1}(2m+1)$ are called *Lin–Wang curves*, and their union is denoted by LW_m. So we have

$$\partial \mathrm{Sph}_{1,1}(2m+1) = LW_m \subset \mathsf{L}_m \subset P(1,2,3).$$

Using our parametrization of L_m we obtain

Theorem. *Lin–Wang curves are real analytic (biholomorphic images of open intervals), and there are $m(m+1)/2$ of them.*

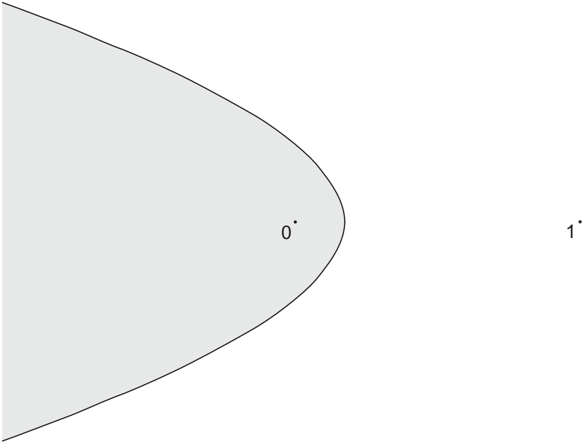
Our parametrization also permits to make pictures of projections of LW_m on the J-plane.

Unsolved problems

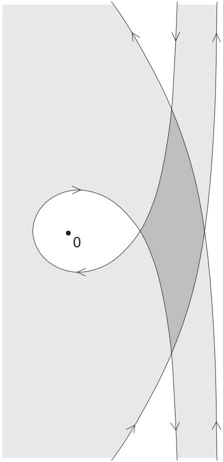
1. What are the critical points of the forgetful map? Are they all simple? Critical values of forgetful maps are zeros if Cohn's polynomials; they are all algebraic since Cohn's polynomials have rational coefficients. Are zeros of Cohn's polynomials simple, except for $J = 0$?

2. Are forgetful maps of spherical surfaces with conic singularities open? At least for tori with one singularity?

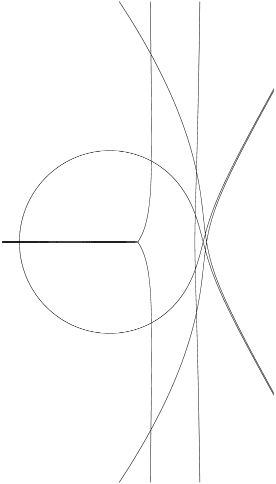
3. Are forgetful maps of spherical surfaces with conic singularities finite-to-one? This is known for tori with four singularities at the points of order 2.



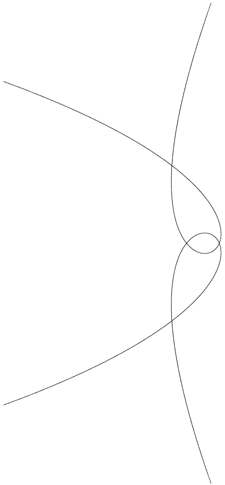
$m = 1$. Projection of $\mathrm{Sph}_{1,1}(3)$ is shaded.



$m = 2$. The boundary curves represent the projection of LW_2, and the shaded region is the *hypothetical* projection of $\mathrm{Sph}_{1,1}(5)$. It is not known that whether the forgetful map is open on $\mathrm{Sph}_{1,1}(2m+1)$.



$m = 3$



Magnification of detail of the previous picture (near the point which looks there as a triple point).

Chiun-Chuan; Chen and Chang-Shou Lin, Mean field equation of Liouville type with singular data: topological degree Comm. Pure Appl. Math. 68 (2015), no. 6, 887–947.

A. Eremenko, G. Mondello and D. Panov, Moduli of spherical tori with one conical point, arXiv:2008.02772.

F. Klein, Forlesungen über die hypergeometrische Funktion, Springer-Verlag, Göttingen, 1933.

G. Lamé, Mémoire sur les axes des surfaces isothermes du second degré considéré comme fonctions de la temperature, J. math. pures appl., IV (1839) 100–125, 126-163.

Robert Maier, Lamé polynomials, hyperelliptic reductions and Lamé band structure, Phil. Trans. R. Soc., A (2008) 366, 1115–1153.

E. T. Whittaker and G. H. Watson, A course of modern analysis, vol. 2, Chapter 23, Cambridge Univ. Press, 1927.