Moduli spaces for spherical tori with one conic singularity

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Every surface can be equipped with a Riemannian metric of constant curvature $-1, 0$ or 1. This metric is unique in each conformal class, except for curvature 0 when it is unique up to scaling.

This 19th century result is equivalent to the Uniformization Theorem.
In this talk we consider only compact orientable surfaces. They are classified by genus $g \geq 0$.
What if we allow simplest singularities of the metric?

Conic singularities

Let D be a sector $\{z : |z| < r, 0 < \arg z < 2\pi\alpha\}$ equipped with a metric of constant curvature $-1, 0$, or 1,

$$ds = \frac{2|dz|}{1 + \kappa|z|^2}, \quad \kappa \in \{-1, 0, 1\}.$$

Gluing the straight boundary segments isometrically we obtain a neighborhood of a **conic singularity** with conic angle $2\pi\alpha > 0$.
(This angle can be arbitrarily large!)
Alternative definition: the length element is

$$ds = \frac{2\alpha|z|^{\alpha-1}|dz|}{1 + \kappa|z|^{2\alpha}}, \quad \kappa \in \{-1, 0, 1\},$$

where z is a conformal coordinate, and κ is the curvature.

Case of non-positive curvature

The Gauss–Bonnet theorem implies

$$\chi(S) + \sum_{j=1}^{n} (\alpha_j - 1) = \frac{1}{2\pi}(\text{total integral curvature})$$

so the expression in the LHS must have the same sign as the curvature κ.

If $\kappa \leq 0$, this is the only obstruction, and if it is satisfied the metric with given n and α_j is unique (up to a constant multiple when $\kappa = 0$) in each conformal class.

This is essentially due to Picard, who devoted to this result 4 long papers in 1893-1931. Proof in modern language was given by M. Heins (1962) and M. Troyanov (1991).

These authors use non-linear PDE and potential theory.

Simple example

A flat metric on the sphere with conic angles α_j exists if and only if

$$2 + \sum_{1}^{n} (\alpha_j - 1) = 0$$

and is unique up to a constant multiple.
If all singularities lie on a circle in the sphere, this circle breaks the sphere into two flat polygons, and the result in this case is nothing but the Schwarz–Christoffel formula.

From now on we consider spherical metrics. For them the problem is wide open.

Moduli spaces of metrics

The first problem is to describe the set of such metrics on a (topological) surface of given genus, with prescribed number of singularities and angles at them.

For $\kappa = 1$, the set of all such metrics is denoted by $\mathrm{Sph}_{g,n}(\alpha_1, \ldots, \alpha_n)$. It has a natural metric: the bi-Lipschitz distance.

One wants to know when this space is not empty, how many connected components it has, what is the topology of those components, and whether these components have any additional natural structure.

Forgetful map

A Riemannian metric defines a conformal structure, and we have the *forgetful map*

$$\pi : \mathrm{Sph}_{g,n}(\alpha_1, \dots, \alpha_n) \to \mathrm{Mod}(g, n),$$

where $\mathrm{Mod}(g, n)$ is the set of conformal structures on surfaces of genus g with n marked points.

The second (much more difficult) problem is to establish the properties of the forgetful map: Is it surjective? If not, how to describe its image?
How many preimages can a point have? In other words, how many such metrics exist in a given conformal class?

Developing map

If S is a surface with a Riemannian metric of constant curvature, with conic singularities at $a_1, \ldots, a_n$, then every smooth point in S has a neighborhood which is isometric to a disk in the sphere, plane or hyperbolic plane. This map is conformal, thus analytic, and we have the multi-valued analytic map from $S \setminus \{a_1, \ldots, a_n\}$ to the hyperbolic plane, or to the Euclidean plane, or to the standard sphere, which is called the *developing map*.
This developing map f is a linearly-polymorphic function:

$$f_\gamma = \phi_\gamma \circ f$$

where $\gamma \mapsto \phi_\gamma$ is a **monodromy** representation of the fundamental group of the punctured surface in the group of linear-fractional transformations. These linear-fractional transformations are hyperbolic (Euclidean, spherical) isometries.

Connection with linear (Fuchsian) ODE

As a linearly-polymorphic function, developing map has a single-valued Schwarzian derivative

$$\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right) = R.$$

At the conic singularities with angles α_j, R has double poles with principal parts $(\alpha_j^2 - 1)/(2z^2)$. Then f is a ratio of two linearly independent solutions $f = w_1/w_2$ of a Fuchsian equation

$$w'' + (R/2)w = 0,$$

where the highest order terms of R at the poles are defined by the angles. A solution of this equation will define a developing map of a spherical metric if the projective monodromy group is conjugate to a subgroup of $PSU(2)$. (For the other two cases, it is a subgroup of the group of Euclidean or hyperbolic motions $SL(2, R)$).

Analogy with work of Klein and Poincare

The coefficient R is determined by the singularities and angles, and $n - 3$ *accessory parameters*.

When there are no singularities, the existence and uniqueness of a metric of constant curvature is equivalent to the Uniformization theorem.

In their early attempts to prove the Uniformization theorem, Klein and Poincare were trying to find accessory parameters of a Fuchsian equation so that the projective monodromy is conjugate to a subgroup of $PSL(2)$.

They encountered serious difficulties, and this approach to the Uniformization theorem has been completed only recently.

The first proofs of the Uniformization theorem were obtained by using the

Connection with non-linear PDE

If $\rho(z)|dz| = e^{u(z)}|dz|$ is a length element of the metric of constant curvature κ expressed in a conformal coordinate, then

$$\Delta u + \kappa e^{2u} = 2\pi \sum_{j=1}^{n} (\alpha_j - 1)\delta_{a_j}.$$

which is the simplest representative of of a class of *mean filed equations* important for mathematical physics. It was studied much for the most important case of the torus with one singularity by Chang-Shou Lin and his school.

What is known?

A. When $\mathrm{Sph}_{g,n}(\alpha_1, \ldots, \alpha_n) \neq \emptyset$? For $g > 0$, the Gauss-Bonnet inequality

$$\chi(S) + \sum_{j=1}^{n} (\alpha_j - 1) > 0$$

is necessary and sufficient. For $g = 0$, we have an additional restriction:

$$d_1(\alpha - 1, \mathrm{Z}_o^n) \geq 1, \qquad (1)$$

where d_1 is the ℓ_1-distance, and Z_o^n is the set of integer points whose sum of coordinates is odd. (Mondello and Panov, 2016). When this inequality is strict, and the Gauss–Bonnet inequality holds, then $\mathrm{Sph}_{g,n}(\alpha_1, \ldots, \alpha_n) \neq \emptyset$.

The case of equality in (1) is special. Monodromy of the developing map in this case is **co-axial** which means isomorphic to a subgroup of the unit circle.

Additional restrictions on the angles of a metric on the sphere apply in this case, and a complete set of conditions was found by Eremenko in 2020.

In the case of co-axial monodromy, each metric with developing map f comes with a continuous family of metrics whose developing maps are of the form $\phi \circ f$, where ϕ is a conformal automorphism of the sphere. Such metrics are called **projectively equivalent**. This family is of real dimension 1 when the monodromy is a non-trivial subgroup of the unit circle and of real dimension 3 for a trivial group.

When the forgetful map is surjective?

A sufficient condition is due to Bartolucci, de Marchis and Malchiodi (2011):

If $g > 0$, $\alpha_j > 1$, $1 \leq j \leq n$,

$$\chi(S) + \sum_{j=1}^{n}(\alpha_j - 1) > 2\min\{\alpha_1, \ldots, \alpha_n, 1\}$$

and

$$\sum_{j=1}^{n} \pm\alpha_j \neq 2k - 2 + n + 2g \qquad (2)$$

for every choice of signs and every $k \in \mathrm{Z}_{\geq 0}$, then the forgetful map is surjective.

Mondello and Panov (2020) proved that under the condition (2) the forgetful map is proper (and thus surjective).

For example, for a torus with one singularity, the forgetful map is proper and surjective if α_1 is not an odd integer.

When α_1 is an odd integer, it is never proper, and for $\alpha_1 = 3$ it is not surjective.

This was generalized by C.-C. Chen and C.-S. Lin, whose result covers the case $g = 0$. Moreover, he computed the topological degree of the forgetful map. Let

$$g(x) = (1 - x)^{n - \chi(S)} \prod_{n=1}^{n} (1 - x^{\alpha_j})$$

$$= 1 + b_1 x^{\beta_1} + b_2 x^{\beta_2} + \dots.$$

If an integer k can be defined so that

$$2\beta_k < \chi(S) + \sum_{j=1}^{n} (\alpha_j - 1) < 2\beta_{k+1}$$

then the degree of the forgetful map is

$$d = \sum_{j=0}^{k} b_j.$$

More known results

1. When S is the sphere, and all $\alpha_j \in (0,1)$ then the necessary and sufficient condition for existence of the metric is

$$0 < 2 + \sum_j (\alpha_j - 1) < 2 \min\{\alpha_j\}$$

and the metric with prescribed angles is unique (F. Luo and G. Tian, 1992).

2. When S is the sphere, and all but at most 3 of the α_j are integers, then the necessary and sufficient condition for existence of the metric is

$$\cos^2 \pi\alpha_1 + \cos^2 \pi\alpha_2 + \cos^2 \pi\alpha_3 + 2(-1)^\sigma \cos \pi\alpha_1 \cos \pi\alpha_2 \cos \pi\alpha_2 < 1,$$

$$\sigma = \sum_{j=4}^{n} (\alpha_j - 1),$$

where it is assumed that $\alpha_4 \ldots, \alpha_n$ are integers. Forgetful map is holomorphic in this case and has degree is $\alpha_4 \cdot \ldots \cdot \alpha_n$. (Eremenko and Tarasov, 2018).

3. If S is a torus with one singularity with angle 6π, then the forgetful map is injective, but not surjective, and its image is explicitly described: it is not dense in the plane $\mathrm{Mod}_{1,1}$ (C.-S. Lin and Wang, 2010, simpler proof: Bergweiler and Eremenko, 2016). These are the only cases when the valence of the forgetful map is known.

4. For tori with one singularity and spheres with 4 singularities, it is known that the forgetful map is finite-to-one, except in the case of coaxial monodromy (Eremenko, 2019). More precisely, for each conformal structure, there are finitely many projective equivalence classes of metrics. The proof is not constructive and gives no explicit upper estimate.

Orbifold notation

In the rest of the talk I describe the results on Problem I (description of the moduli space) in the simplest case of a torus with one singularity.

We will describe $\mathrm{Sph}_{1,1}(\alpha)$ (which is a surface). It is convenient to use the notion of orbifold.
An 2-dimensional orbifold is as compact surface S equipped with a function $n : S \to \mathsf{Z}_{\geq 1} \cup \{+\infty\}$ such that $n(x) = 1$ for all but finitely many points. $n(x) = \infty$ is interpreted as a puncture. The points where $n(x) > 1$ are called orbifold points. The orbifold Euler characteristic is defined as

$$\chi^O = \chi(S) - \sum \left(1 - \frac{1}{n(x)}\right),$$

where $\chi(S) = 2 - 2g$ is the usual Euler characteristic.

Main new results

Description of $\mathrm{Sph}_{1,1}(\alpha)$ where α is not an odd integer.
Let $m = [(\alpha+1)/2]$, so that $2m$ is the closest even integer to α. Then $\mathrm{Sph}_{1,1}(\alpha)$ is a connected surface of genus $[(m^2 - 6m + 12)/12]$ with m punctures.

It has a natural orbifold structure with one orbifold point of order 3 when $d_1(\alpha, 6\mathrm{Z}) > 1$ and one orbifold point of order 2 when $d_1(\alpha, 2\mathrm{Z}) > 1$. The orbifold Euler characteristic is

$$-m^2/6.$$

When $\alpha = 2m$ is an even integer, $\mathrm{Sph}_{1,1}(\alpha)$ has a natural conformal structure such that the forgetful map is complex analytic. The Riemann surface $\mathrm{Sph}_{1,1}(\alpha)$ is a Belyi curve.

Description of $\mathrm{Sph}_{1,1}(2m+1)$ for $m \in \mathbb{Z}_{>0}$.
$\mathrm{Sph}_{1,1}(2m+1)$ is a 3-dimensional manifold which consists of $\lfloor m(m+1)/6 \rfloor$ connected components. Each component is homeomorphic $D \times \mathbb{R}$, where D is an open disk.

Each projective equivalence class contains one metric which is invariant with respect to conformal involution of the torus. The space of equivalence classes has a natural orbifold structure: it consists of $\lfloor m(m+1)/6 \rfloor$ disks. When $m \equiv 1 \pmod 3$, one of these disks has an orbifold point of order 3, and there are no other orbifold points.

The method of the proof consists in partition of the spherical torus with one conic singularity into two spherical triangles.
A spherical triangle with angles $(\alpha_1, \alpha_2, \alpha_3)$ is called *balanced* if

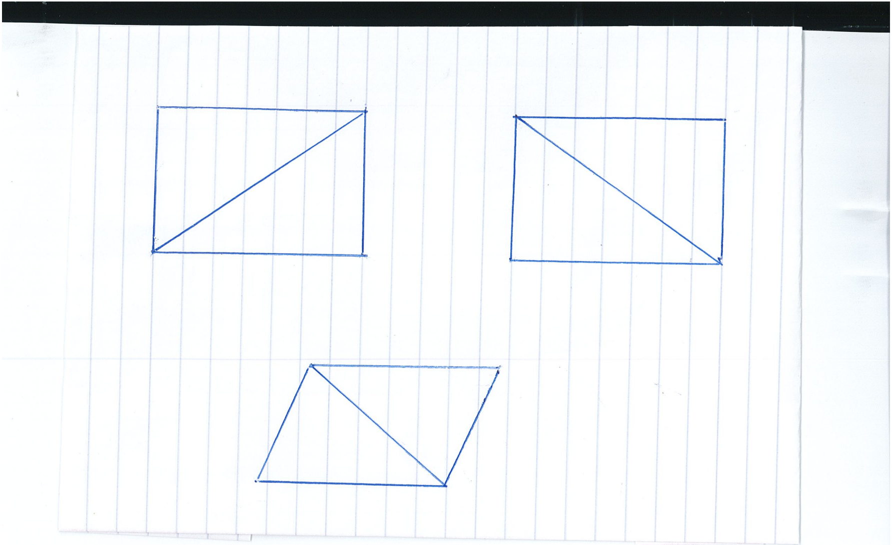
$$\alpha_i \leq \alpha_i + \alpha_k$$

for all permutations (i, j, k) of $(1, 2, 3)$. It is *strictly balanced* if all these inequalities are strict, and *marginal* if at least one of them is an equality.
Theorem. *Every spherical torus with a conic singularity with angle* $\alpha > 1$ *is a union of two congruent balanced spherical triangles with disjoint interiors and angles* $\alpha_1, \alpha_2, \alpha_3$ *such that*

$$\alpha = 2(\alpha_1 + \alpha_2 + \alpha_3).$$

This decomposition is unique if the triangles are strictly balanced, and there are at most two such decompositions when they are marginal.

Example: a flat torus with no singularity ($\alpha = 1$):



Angles of spherical triangles

1. If three positive numbers $\alpha_1, \alpha_2, \alpha_3$ are not integers, then there is a spherical triangle with angles $\pi\alpha_j$ iff

$$\cos^2 \pi\alpha_1 + \cos^2 \pi\alpha_2 + \cos^2 \pi\alpha_3 + 2 \cos \pi\alpha_1 \cos \pi\alpha_2 \cos \pi\alpha_2 < 1.$$

A triangle with such angles is unique.
2. If exactly one, say α_i is an integer, then a triangle with angles $(\pi\alpha_1, \pi\alpha_2, \pi\alpha_3)$ exists iff wither $\alpha_j + \alpha_k$ or $|\alpha_j - \alpha_k|$ is an integer m of the opposite parity to α_i, and

$$m \leq \alpha_i - 1.$$

For any such angles, a one-parametric family of triangles with these angles exists.
3. If two of the $(\alpha_1, \alpha_2, \alpha_3)$ are integers, then all three are integers and a triangle with angles $\pi\alpha_j$ exists iff $\alpha_1 + \alpha_2 + \alpha_3$ is odd, and

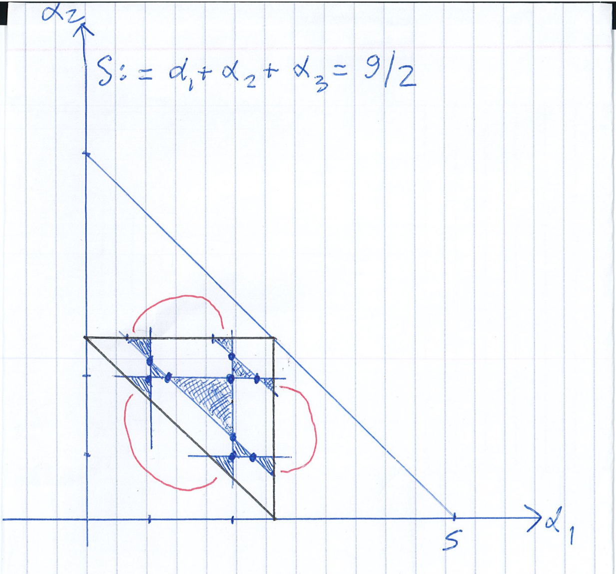
$$\max\{\alpha_2, \alpha_2, \alpha_3\} \leq (\alpha_1 + \alpha_2 + \alpha_3 - 1)/2.$$

Condition for non-integer α_j can be rewritten as

$$\cos^2 \pi\alpha_1 + \cos^2 \pi\alpha_2 + \cos^2 \pi\alpha_3 + 2\cos \pi\alpha_1 \cos \pi\alpha_2 \cos \pi\alpha_2$$

$$\begin{aligned} = \quad & \cos \pi \frac{\alpha_1 + \alpha_2 + \alpha_3}{2} \cos \pi \frac{\alpha_1 + \alpha_2 - \alpha_1}{2} \\ \times \quad & \cos \pi \frac{\alpha_1 - \alpha_2 + \alpha_3}{2} \cos \pi \frac{-\alpha_1 + \alpha_2 + \alpha_3}{2} < 0. \end{aligned}$$

For the fixed sum $\alpha_1 + \alpha_2 + \alpha_3$ this defines a set consisting of triangles which are shaded in the following pictures.



α_2
$S := \alpha_1 + \alpha_2 + \alpha_3 = 9/2$
S
α_1

Embedding to the moduli spaces of Lamé equations

Developing map f of a spherical torus with one conic singularity is a ratio of two linearly independent solutions of Lamé equation: $f = w_1/w_2$

$$w'' = \left(\frac{\alpha^2 - 1}{4}\wp + \lambda\right) w,$$

where $\wp$ is the Weierstrass function of the torus, and λ is a complex *accessory parameter*.

This Lamé equation defines a developing map of a spherical torus if and only if the projective monodromy group is a subgroup of $PSU(2)$.

Lamé equation depends on parameters $\alpha, g_2, g_3, \lambda$, where g_2 and g_3 are the lattice invariants:

$$g_2 = 60 \sum_{\omega \neq 0} \omega^{-4}, \quad g_3 = 140 \sum_{\omega \neq 0} \omega^{-6},$$

where the summation is over elements of the lattice.

We fix real α and call two Lamé equations *equivalent* if they are obtained from each other by the change of the independent variable $z \mapsto kz$ (scaling of the lattice).
This gives

$$(g_2, g_3, \lambda) \sim (k^4 g_2, k^6 g_3, k^2 \lambda).$$

The set of equivalence classes is called the *moduli space of Lamé equations.* It is a weighed projective plane $P(1, 2, 3)$.
The condition that the monodromy is in $PSU(2)$ gives one complex equation on λ and the parameter of the torus but this equation is not complex analytic unless α is an even integer. So our moduli spaces $\mathrm{Sph}_{1,1}(\alpha)$ are 2-dimensional surfaces in $P(1, 2, 3)$.
When α is an even integer, $\mathrm{Sph}_{1,1}(\alpha)$ is an algebraic curve in $P(1, 2, 3)$.

These surfaces are properly embedded (have no boundary) except for the case when α is an odd integer. This interesting case corresponds to the classical Lamé equation

$$w'' = ((m(m+1)\wp + \lambda)\, w, \quad m \in \mathrm{Z}_{\geq 1}. \tag{3}$$

which was studied by Lamé himself. A solution w of (3) is called a Lamé function if w^2 is a polynomial. Such a solution exists iff an algebraic equation

$$F_m(g_2, g_3, \lambda) = 0$$

is satisfied. Here g_2, g_3 are the fundamental invariants of the lattice associated to our torus. To each Lamé function an elliptic differential of the second kind is: $\omega = w^{-2}(z)dz$. This differential has vanishing residues, so it has two periods corresponding to the generators of the homology group of the torus.

Let LW_m be the set of Lamé equations which posses a Lamé function w as a solution, such that the ratio of the periods of the differential $w^{-2}dz$ is real. Then we have the following description of the boundary of our moduli space:

$$\partial \mathrm{Sph}_{1,1}(2m+1) = LW_m.$$

So the boundary of our real-analytic surface $\mathrm{Sph}_{1,1}(2m+1)$ is a real analytic curve which belongs to a complex algebraic curve of Lamé equations which possess a Lamé function.

The moduli space of Lamé functions

It is isomorphic to the space of pairs

$$\mathsf{L}_m := (\text{elliptic curve}, \text{Abelian differential}),$$

where the differential has a single zero of multiplicity $2m$ at the origin and all poles double, with zero residues.

Such differentials define a *singular Euclidean metric* on the torus, with one conic singularity at the origin and several poles isometric to $\{z : r < |z| < \infty\}$.
A study of such singular flat metrics is possible with the same methods that we employed for spherical metrics. The key fact is that every flat singular torus with one conic singularity is the union of two congruent flat singular triangles, and this decomposition is almost unique.
This permits to describe the moduli space of Lamé functions.

Denote

$$d_m^I = \begin{cases} m/2+1, & m \equiv 0 \pmod 2, \\ (m-1)/2, & m \equiv 1 \pmod 2. \end{cases}$$

$$d_m^{II} = 2\lceil m/2 \rceil.$$

Theorem. *When $m \geq 1$, L_m is a Riemann surface which consists of two connected components L_m^I and L_m^{II}.*
It has a natural orbifold structure with one orbifold point of order 3 on L_m^I when $m \not\equiv 1 \pmod 3$, and one orbifold point of order 2 which is in L_m^I when $m \in \{2,3\} \pmod 4$ and in L_m^{II} otherwise. Component L_m^I has d_m^I punctures, and component L_m^{II} has $2d_m^{II}/3$ punctures. The orbifold Euler characteristics are

$$\chi^O(\mathsf{L}_m^I) = -(d_m^I)^2/6, \quad \chi^O(\mathsf{L}_m^{II}) = -(d_m^{II})^2/18.$$

Main unsolved questions.

1. Is the forgetful map open? This is unlikely in general, but perhaps it is open for the case of tori with one singularity.

2. What is the maximum number of preimages under the forgetful map? It is only known that it is finite, for tori with one singularity. No upper estimate is known. It is conjectured that when α is not an odd integer, the maximum valence is equal to its topological degree which was computed by C. C. Chen and C. S. Lin.
The only case when the answer to this question is known is $m = 1$ when the forgetful map is injective.

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