Real solutions of Painlevé VI and special pentagons

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Richard Fuchs was a son of Lazarus Fuchs. Father Fuchs is famous for Fuchsian groups, and several (at least three different kinds of) "Fuchs conditions" in the analytic theory of differential equations.

Fuchsian differential equations

A singular point a of a linear ODE

$$w'' + p(z)w' + q(z)w = 0,$$

where p and q are rational functions, is called **regular** if all solutions satisfy $w(z) = O(z-a)^K$ near a, for some real K.
The necessary and sufficient condition for this (Lazarus Fuchs) is

$$p(z) = O(z-a)^{-1} \text{ and } q(z) = O(z-a)^{-2}.$$

In this case there are fundamental solutions of the form

$$w_1(z) = (z-a)^{\rho_1}(1+g_1(z)), \quad w_2(z) = (z-a)^{\rho_2}(1+g_2(z)),$$

where g_j are holomorphic at a, $g_j(a) = 0$, ρ_j are **exponents** at a, solutions of the **indicial equation**: $\rho(\rho-1) + p_0\rho + q_0 = 0$.
We assume that $\rho_1 - \rho_2 > 0$ is **not an integer**.

If all singular points are regular, the equation is called **Fuchsian**.

To describe the global behavior of solutions of a Fuchsian ODE, one uses the **projective monodromy** representation

$$M : \pi_1(\overline{\mathbf{C}}\backslash\text{singularities}) \to PSL(2, \mathbf{C}).$$

Question: Which monodromy representations can occur?

Riemann solved this question completely when the number of singularities $n = 3$.

When $n > 3$, parameter count (Henri Poincaré) shows that an equation with prescribed singularities and prescribed monodromy does not exist. But if one introduces **apparent singularities** (with trivial monodromy) then the number of parameters matches.

The simplest case occurs when the number of non-apparent singularities $n = 4$ and one singularity is apparent.

Richard Fuchs studied in 1905 the following differential equation:

$$w'' - \left(\frac{1}{z-q} + \sum_{j=1}^{3} \frac{\kappa_j - 1}{z - t_j} \right) w' + \left(\frac{p}{z-q} - \sum_{j=1}^{3} \frac{h_j}{z - t_j} \right) w = 0$$

with five singularities at $(t_1, t_2, t_3, t_4, q) := (0, 1, x, \infty, q)$.

The singularities at t_j have exponents $\{0, \kappa_j\}$, for $1 \leq j \leq 3$, and the exponents at q are $\{0, 2\}$.

Richard Fuchs imposed the following conditions:

a) the singularity at ∞ is regular, and has exponent difference κ_4,

b) the singularity at q is apparent (has trivial monodromy).

For given κ_j, $1 \leq j \leq 4$, and given p, q, x, these conditions determine parameters h_j uniquely.

Suppose that all κ_j are fixed, and let us move x continuously.

Question: How should $p(x), q(x)$ change so that the monodromy of this equation remains unchanged?

Answer: $q(x)$ must satisfy the following **non-linear** ODE:

$$q_{xx} = \frac{1}{2}\left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-x}\right) q_x^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{q-x}\right) q_x$$
$$+ \frac{q(q-1)(q-x)}{2x^2(x-1)^2}\left\{\kappa_4^2 - \kappa_1^2 \frac{x}{q^2} + \kappa_2^2 \frac{x-1}{(q-1)^2} + (1-\kappa_3^2)\frac{x(x-1)}{(q-x)^2}\right\},$$

which is called the **Painlevé VI** equation.

This was the first example of an "isomonodromic deformation".

Equation PVI was rediscovered by Painlevé and Gambier, who were solving a different problem: to find all equations of the form

$$y'' = R(y', y, z), \quad \text{where } R \text{ is rational},$$

without **movable singularities**.

PAUL PAINLEVÉ

BERTRAND GAMBIER

A singularity (other than a pole) of a solution of a nonlinear ODE is **movable** when its location depends on initial conditions. Solutions of linear ODE never have movable singularities.

The classification of nonlinear ODE without movable singularities consists of about 50 types of equations. Most of them can be solved in terms of the classical special functions (satisfying either linear or first order nonlinear ODE).

Six equations of this classification define new functions, which cannot be expressed in terms of the classical special functions. These are the six **Painlevé equations**.

The sixth one, **PVI**, is the most general, in the sense that all others can be obtained from it by the certain "confluency" process.

All solutions of PVI are ramified only at its **fixed singularities** $x = 0, 1, \infty$.

The conditions of Cauchy's theorem for a solution $q(x)$ are violated at the points where $q(x) \in \{0, 1, x, \infty\}$. These points x are either removable singularities or poles of q. We call them **special points**.

We consider **real** solutions $q(x)$ of PVI with **real** parameters, on an interval of the real line between two adjacent fixed singularities $0, 1, \infty$. WLOG we can choose the interval $(1, \infty)$.

We will explain geometric interpretation of these solutions, and obtain an algorithm which determines the number and mutual position of special points on the interval.

More precisely, the outcome of the algorithm is a sequence of symbols $\{0, 1, x, \infty\}$ which shows the order in which special points of a solution appear on the segment $(1, \infty)$.

Real Fuchsian ODE and circular polygons

A linear Fuchsian ODE

$$w'' + p(z)w' + q(z)w = 0$$

with all real parameters (coefficients, singularities and exponents) is called **real**.

The ratio $f = w_1/w_2$ of two linearly independent solutions is meromorphic and locally univalent in the upper half-plane H.

At a singular point t,

$$f(z) = f(t) + (c + o(1))(z - t)^\kappa,$$

where $\kappa > 0$ is the exponent difference at t.

If $\kappa = 0$ but the singularity at t is not apparent, then

$$f(z) = f(t) + (c + o(1))/\log(z - t).$$

The function f maps each interval (t_{j-1}, t_j) between singular points into a circle C_j on the Riemann sphere, and has conical singularities at the singular points t_j.
Such functions are called **developing maps** (of circular polygons).

The formal definition of a **circular polygon** is

$$Q = (\overline{D}, t_1, \ldots, t_n, f),$$

where $\overline{D}$ is a closed disk, $t_j \in \partial D$ are distinct boundary points, and f is a developing map with conical singularities at t_j.

The intervals (t_{j-1}, t_j) are called the **sides** of Q, the points t_j are its **corners**, and κ_j are the interior **angles** at the corners.

We measure all angles in half-turns instead of radians!

Two circular n-gons

$$Q = (\overline{D}, t_1, \ldots, t_n, f) \text{ and } Q' = (\overline{D'}, t'_1, \ldots, t'_n, f_1)$$

are **equal** if there is a conformal homeomorphism $\phi : \overline{D'} \to \overline{D}$ such that $\phi(t'_j) = t_j$ and

$$f_1 = f \circ \phi. \tag{1}$$

Two circular n-gons are **equivalent** if instead of (1) we require only $f_1 = L \circ f \circ \phi$, with some linear-fractional transformation L.

For polygons which are subsets of the sphere this means that one can be mapped onto another by a linear-fractional transformation.

There is a one-to-one correspondence between equivalence classes of circular polygons and normalized real Fuchsian equations.
The developing map of a circular polygon is the ratio of two linearly independent solutions of the corresponding Fuchsian equation.

This fact was known to Schwarz, and possibly to Riemann.

ÉMILE PICARD

Equation of Richard Fuchs and special pentagons

$$w'' - \left(\frac{1}{z-q} + \sum_{j=1}^{3} \frac{\kappa_j - 1}{z - t_j} \right) w' + \left(\frac{p}{z-q} - \sum_{j=1}^{3} \frac{h_j}{z - t_j} \right) w = 0$$

This equation with five singularities defines a circular pentagon. But one singularity q is special: it has exponents $0, 2$ and trivial monodromy.

We say in such case that our circular pentagon Q has a **slit**, and call $f(q)$ the **tip** of the slit.

Such pentagons (with one slit) are called **special pentagons**.

There is a one-to-one correspondence between real normalized Fuchsian equations with 5 singularities, one of them apparent with exponent difference 2*, and equivalence classes of special pentagons.*

Isomonodromic deformation of special pentagons

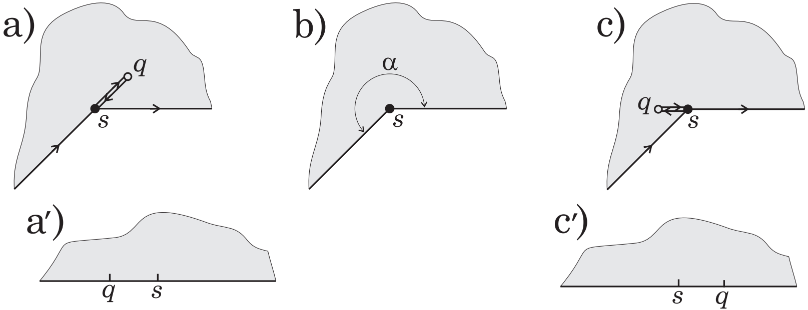
The sides of a special pentagon are mapped into four circles which are recovered from the monodromy. It can be deformed, with the four circles fixed, only by moving the tip of the slit along its circle.

The parameters x and q are functions of the tip position, x is a monotone function of the tip, and $q(x)$ is a solution of PVI. We denote our special pentagon by Q_x.

When the slit shortens, it eventually vanishes, and q collides with either one or two corners of Q_x. If q collides with one corner as $x \to x_0$, then Q_{x_0} is a quadrilateral, and x_0 is a **special point**. Otherwise, Q_x **degenerates** to a triangle, and $x \to 1$ or $x \to \infty$.

When the slit lengthens, it eventually hits the boundary of Q_x from inside, splits Q_x into two parts, and becomes a **cross-cut**. If one of the two parts is a quadrilateral then x_0 is a **special point**. Otherwise Q_x **degenerates** as $x \to x_0$.

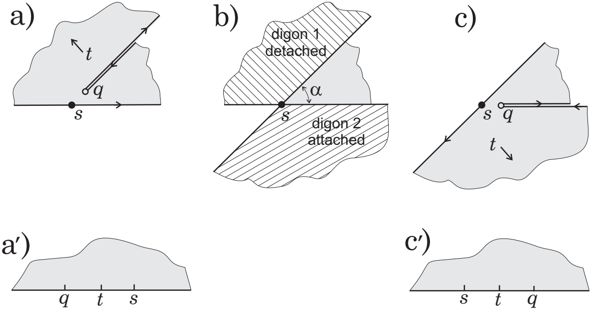
What happens to Q_x when x passes beyond x_0?



Transformation 1. Let $q \in (t_{k-1}, t_k)$ and the slit **shortens**. As the slit vanishes, q collides with s, which is either t_k or t_{k-1}. When $q = s$, our special pentagon becomes a quadrilateral.

As x passes s we get a special pentagon with the sides mapped to the same four circles, but q and s exchange their order.

The slit **lengthens** now. If it was on C_k, it is now on C_{k+1} if $s = t_k$, and on C_{k-1} if $s = t_{k-1}$.

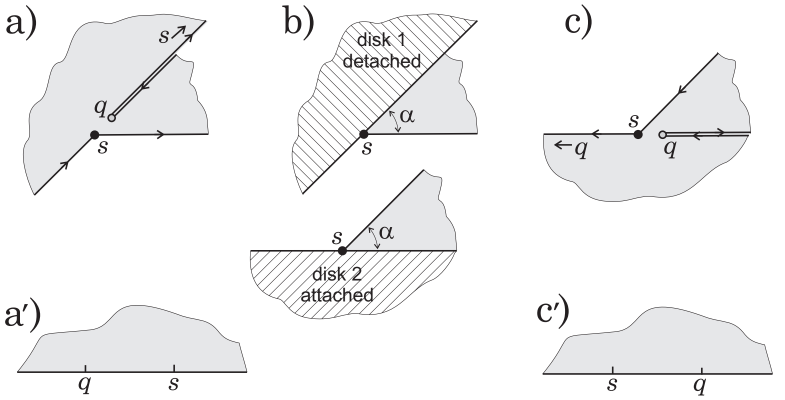


Transformation 2. Let $q \in (t_{k-1}, t_k)$, and the slit **lengthens**. When it hits the boundary of Q_x from inside, it becomes a **cross-cut**, splitting the pentagon into two parts. Let s be the point where this happens: $f(q) \to f(s)$ as the slit lengthens.

A digon with corners at t and s is detached in the limit.

After collision, a new digon is attached. It has the same angle as the old one, and is bounded by the same two circles.
We call it the **vertical digon** to the old one.

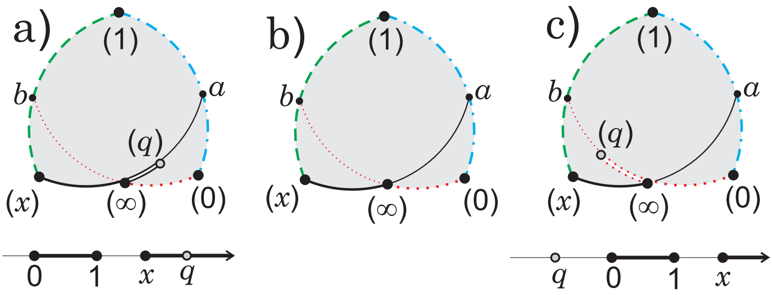
The three points q, t, s exchange their order.



Transformation 3. When the slit lengthens, hits the boundary of Q_x from inside, and the special pentagon splits, as in Transformation 2, assume now that the slit hits a corner s. If $q \in (t_{k-1}, t_k)$ then $f(q) \to f(s)$, where s is either t_{k-1} or t_k. This means that a disk (rather than digon) detaches in the limit, and a new disk is attached as x passes x_0.

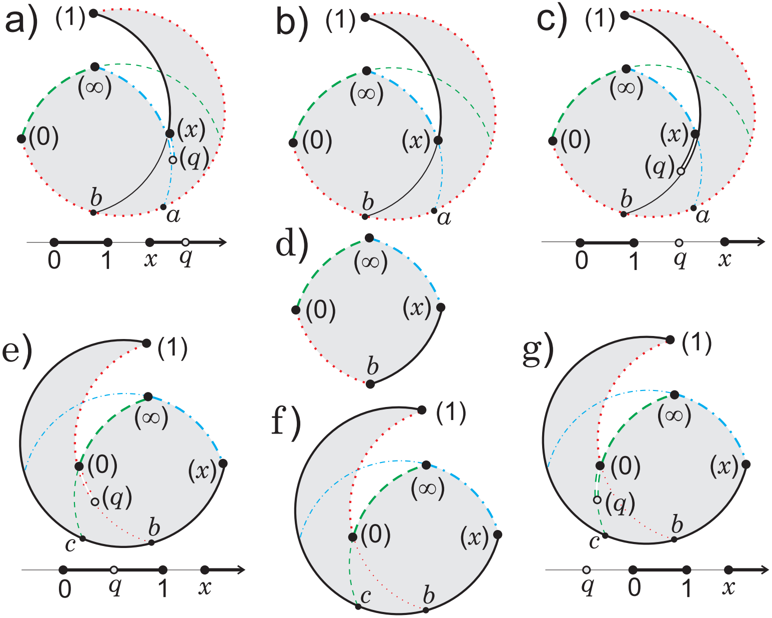
The points q and s exchange their order.

Examples



If the slit in (a) lengthens and hits the boundary, the special pentagon splits and degenerates, as $x \to 1$. If the slit in (a) shortens and vanishes, the special pentagon becomes a quadrilateral in (b), where $q(x) = \infty$.
Then a new slit grows in (c). When it hits the boundary, the special pentagon splits and degenerates, as $x \to +\infty$.

The solution $q(x)$ has only one special point, a pole, on $(1, +\infty)$. There is one Transformation 1 in this example.



a)
(1)
(∞)
(0)
(x)
(q)
b
a
0
1
x
q
b)
(1)
(∞)
(0)
(x)
b
a
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(1)
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b
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0
1
x

If the slit in (a) lengthens and hits the boundary, the special pentagon degenerates, as $x \to 1$. If the slit in (a) shortens and vanishes, there is Transformation 1 in (b), where the special pentagon becomes a quadrilateral.

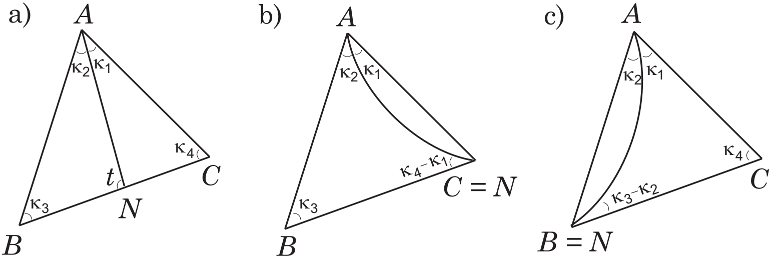
A new slit grows in (c). When it hits the boundary, there is Transformation 2 in (d): a digon is detached and its vertical digon is attached.

A new slit shortens in (e). When it vanishes, there is Transformation 1 in (f), where the special pentagon becomes a quadrilateral.

A new slit grows in (g). When it hits the boundary, the special pentagon degenerates, as $x \to \infty$.

The solution $q(x)$ has three special points: $x_0 < x_1 < x_2$ with $q(x_0) = x_0,\ q(x_1) = 1,\ q(x_2) = 0.$

Solutions without special points on $(1, +\infty)$



If $q(x)$ does not have special points on $(1, +\infty)$ then Q_x should degenerate both when it shortens and vanishes and when it lengthens and hits the boundary.

This implies that our special pentagon is a circular triangle ABC with the slit Aq on a circular arc AN with one end at A and another end at a point N on the side BC opposite A, either inside AB as in (a) or at a corner as in (b) and (c).

Theorem. *A real solution $q(x)$ of PVI satisfying $0 < q(x) < 1$ for $1 < x < +\infty$ exists for arbitrary real parameters κ_j.*

Two of the three generators of the monodromy representation of the corresponding linear Fuchsian equation have a common fixed point with the multipliers $e^{2\pi i\kappa_1}$ and $e^{2\pi i\kappa_2}$.

Depending on the values of κ_j, there is either one such isolated solution, or a whole interval of them.

For the special values of parameters $(1/2, 1/2, 1/2, 1/2)$ and $(1/2, 1/2, 1/2, 3/2)$ this was recently proved by Z-J. Chen, T.-J. Kuo and C.-S. Lin.

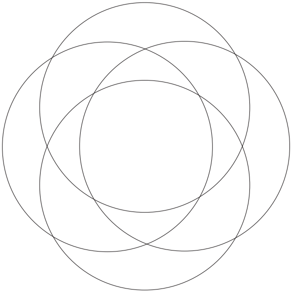
Representation of polygons by nets

Let $Q = (\overline{D}, t_1, \ldots, t_n, f)$ be a circular n-gon. Let C_j be the circle containing $f((t_{j-1}, t_j))$. Circles C_j define a cell decomposition of the sphere which we call the *lower configuration*. The f-preimage of the lower configuration is a cell decomposition of the closed disk $\overline{D}$ which is called the **net** of our polygon. Vertices of the net at the corners are labeled by t_j.

Two nets are equivalent if there is an orientation-preserving homeomorphism of $\overline{D}$ sending one net to another and labeled vertices to similarly labeled vertices.
Specifying the cells

$$(f(t_1),\ f(e),\ f(T))$$

of the lower configuration defines the polygon uniquely.
So a polygon is completely determined by the lower configuration, the net and the normalization data.
It is difficult to describe intrinsically all possible nets on a given lower configuration. But in the case when $n = 4$ and the lower configuration is homeomorphic to a generic quadruple of great circles, one can give such an intrinsic description.



The corresponding cell decomposition of the sphere has the following property:

a) *any pair of* 2*-cells adjacent along a* 1*-cell consists of a triangle and a quadrilateral.*

This property is inherited by the net. Two additional property of the net are:

b) *every interior vertex has degree* 4, *and every vertex on a side has degree 3*, and
c) *the degrees of the corners (as vertices of the net) are even.*
The last property follows from our assumption that the circles C_j and C_{j+1} are distinct.
One can show that these three properties a), b) and c) characterize the nets over lower configurations homeomorphic to generic configurations of 4 great circles.
This permits to construct many examples of nets, circular quadrilaterals and special circular pentagons.

Transformations 1, 2, 3 above can be explicitly performed on the nets.

Lower configurations of four great circles correspond to $PSU(2)$ monodromy representations.

Properties of special points of real PVI solutions strongly depend on the topological type of the lower configuration. For example, *The number of special points is infinite if and only if some two circles of the lower configuration are disjoint. It is infinite in one direction if there is one pair of disjoint circles, and in both directions if there are two such pairs.*

Classification of generic configurations

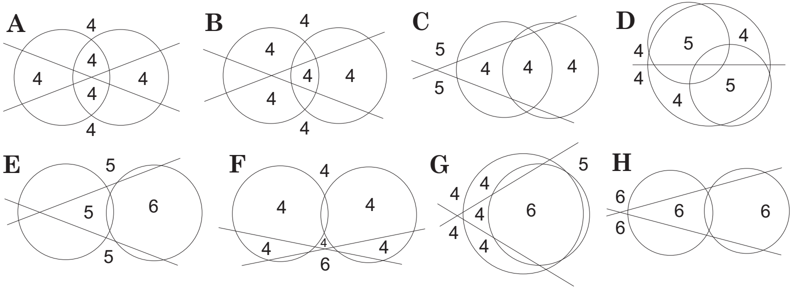


Figure: Generic chains A-H. All pairs of circles intersect.

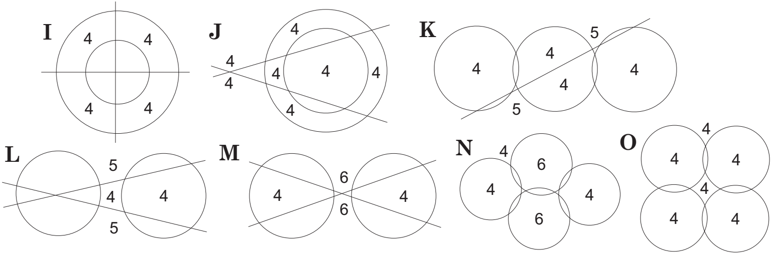


Figure: Generic chains I-O. Some pairs are disjoint.