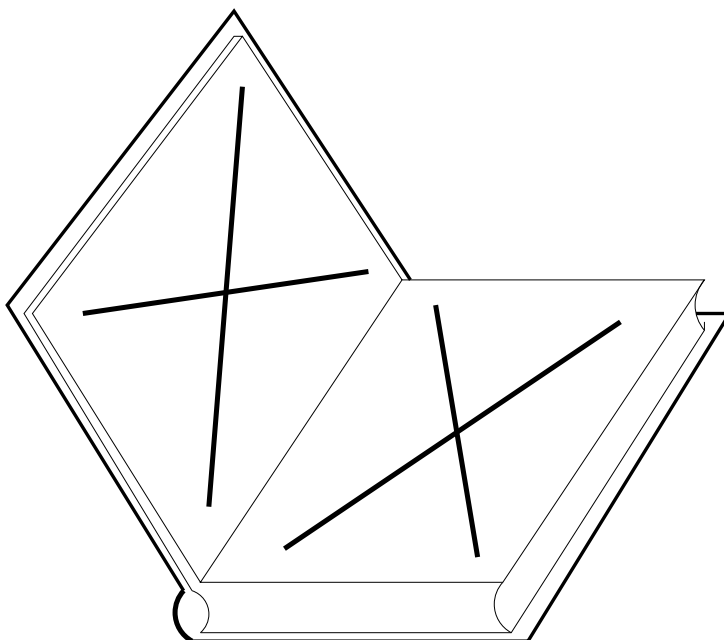# THE WRONSKI MAP

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1. Enumerative geometry

Given 4 lines in general position in 3-space, how many lines intersect all of them?

Answer: 2.

H. Schubert, Kalkül der abzählenden Geometrie

1879

Problem: *Given mp subspaces of dimension p in general position in a vector space of dimension $m + p$, how many subspaces of dimension m intersect all these given subspaces nontrivially?*

Let the row spaces of

$$Q_j \in \mathsf{Mat}(p \times (m + p)), \quad j = 1, \ldots, mp$$

represent the given subspaces, and the row spaces of $K \in \mathsf{Mat}(m \times (m + p))$ the unknown subspaces. Then the condition is equivalent to

$$\det \begin{vmatrix} Q_j \\ K \end{vmatrix} = 0, \quad j = 1, \ldots, mp.$$

Two solutions K_1 and K_2 are considered equivalent if $K_1 = AK_2$ for $A \in GL(m)$.

How many equivalence classes of solutions does this system have?

Answer for complex spaces (Schubert, 1886):

$$d(m,p) = \frac{1!\,2!\ldots(p-1)!\,(mp)!}{m!(m+1)!\ldots(m+p-1)!}.$$

For example:

$m =$	2	3	4	5	6
$p = 2$	2	5	14	42	132
$p = 3$		42	462	6006	87516
$p = 4$			24024	1662804	140229804

Notice: $d(m,2)$ is the m-th Catalan number.

We study this problem for **real** subspaces. For example, if $(m,p) = (2,2)$, it may have no real solutions.

Combinatorial interpretation: $d(m, p)$ is the number of Standard Young Tableaux of the shape $p \times m$.

1	3	4	9
2	6	7	11
5	8	10	12

Or the number of **ballot sequences**: mp electors vote for p candidates, $a, b, c, \ldots$, such that at any time a leads over b, b leads over c etc., and the election ends in a tie between all candidates. For example,

$$a, b, a, a, c, b, b, c, a, c, b, c.$$

2. Control of a linear system by static output feedback.
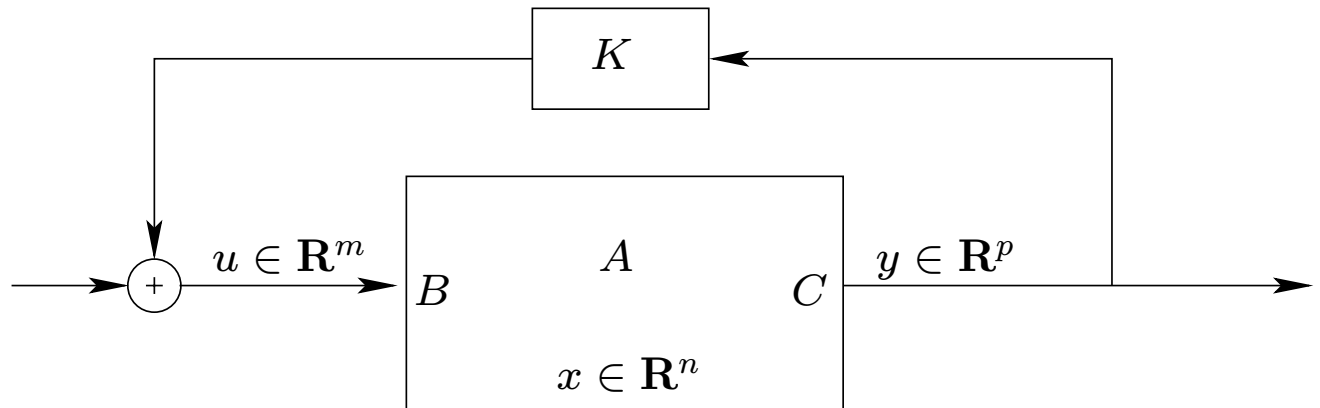
$$\begin{aligned}
\dot{x} &= Ax + Bu, \\
y &= Cx, \\
u &= Ky.
\end{aligned}$$

Elimination gives

$$\dot{x} = (A + BKC)x.$$

Pole placement Problem: *given real A, B, C and a real polynomial q of degree n, find real K, so that*

$$\det(\lambda I - A - BKC) = q(\lambda).$$

Using:

a) A coprime factorization

$$C(zI - A)^{-1}B = D(z)^{-1}N(z),$$

$$\det D(z) = \det (zI - A),$$

b) The identity

$$\det (I + PQ) = \det (I + QP),$$

we rewrite the characteristic polynomial as

$$\det (zI - A - BKC)$$
$$= \det (zI - A)\det (I - (zI - A)^{-1}BKC)$$
$$= \det (zI - A)\det (I - C(zI - A)^{-1}BK)$$
$$= \det D(z)\det (I - D(z)^{-1}N(z)K)$$
$$= \det (D(z) - N(z)K)$$
$$= \begin{vmatrix} D(z) & N(z) \\ K & I \end{vmatrix}.$$

We want to find a **real** K, so that this determinant has prescribed zeros z_j:

$$\begin{vmatrix} D(z_j) & N(z_j) \\ K & I \end{vmatrix} = 0, \quad j = 1, \ldots, n.$$

These are n equations in mp variables (the entries of K).

If $n > mp$ the problem is unsolvable for an open set of data.

If $n < mp$ it always has (real!) solutions (A. Wang, 1996).

The critical case: $n = mp$. This is a special case of Schubert's problem.

3. Rational curves with prescribed inflection points

$$f : \mathbf{P}^1 \to \mathbf{P}^n, \quad f = (f_1 : \ldots : f_p), \quad p = n + 1.$$

A point $z \in \mathbf{P}^1$ is an inflection point if the Wronskian determinant

$$W(f_1, \ldots, f_p) = \begin{vmatrix} f_1 & \cdots & f_1^{p-1} \\ f_2 & \cdots & f_2^{p-1} \\ & \cdots & \\ f_p & \cdots & f_p^{p-1} \end{vmatrix}$$

equals zero at this point.

A real rational curve is called **maximally inflected** if all inflection points are real.

If $d = \deg f$ we have $f = Ae$, where

$$e(z) = (1, z, \ldots, z^d), \quad E : \mathbf{P}^1 \to \mathbf{P}^d,$$

and A is a linear projection $\mathbf{P}^d \to \mathbf{P}^n$.

Then

$$W(f_1, \ldots, f_p) = \det AE,$$

where

$$E(z) = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ z & 1 & \ldots & 0 \\ z^2 & 2z & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ z^d & dz^{d-1} & \ldots & (z^d)^{(n)} \end{pmatrix},$$

the matrix whose columns are derivatives of $e(z)$. So $W(z) = 0$ if and only if $\mathrm{Ker} A$ intersects $\mathrm{Im} E$ non-trivially.

Thus the problem of finding maximally inflected curves with prescribed inflection points is a special case of the Schubert problem and of the Pole placement problem.

4. Grassmannian $G(m, m + p)$
= the set of m-subspaces in $\mathbf{C}^{m+p}$
= the set of $m \times (m + p)$ matrices of rank m, modulo equivalence:

$A \sim B$ if $A = UB$, $\det U \neq 0$.

Plücker embedding:

$$G(m, m + p) \to \mathbf{P}^N, \ N = \binom{m + p}{m} - 1.$$

All $m \times m$ minors serve as homogeneous coordinates.

An m-subspace K intersects a given p-subspace Q non-trivially iff

$$\det \begin{vmatrix} Q \\ K \end{vmatrix} = 0,$$

a *linear* equation in Plücker's coordinates. $d(m, p)$ is the number of intersections of $G_{\mathbf{C}}(m, m + p)$ with a generic subspace of codimension mp in $\mathbf{CP}^N$. It does not depend on the subspace in the complex case.

5. The Wronski map $\phi : G(m, m + p) \to \mathbf{P}^{mp}$,

$$(q_1, \ldots, q_p) \mapsto W(q_1, \ldots, q_p).$$

Here a point in $G(m, m + p)$, that is an m-subspace in $\mathbf{C}^{m+p}$, is represented by p polynomials of degree $d = m + p - 1$, their coefficients serve as coefficients of p linear forms defining the m-subspace. The Wronskian determinant is a polynomial of degree $mp + 1$, it is identified with a point in $\mathbf{P}^{mp}$, using the coefficients as homogeneous coordinates of this point.

ϕ is a regular map between smooth projective varieties. It is in fact a projection map, restricted to the image of the Grassmannian under the Plücker embedding.

6. Degree of a map $f : X \to Y$.

a) If X is oriented,

$$\deg f = \pm \sum_{x \in f^{-1}(y)} \operatorname{sgn} \det f'(x).$$
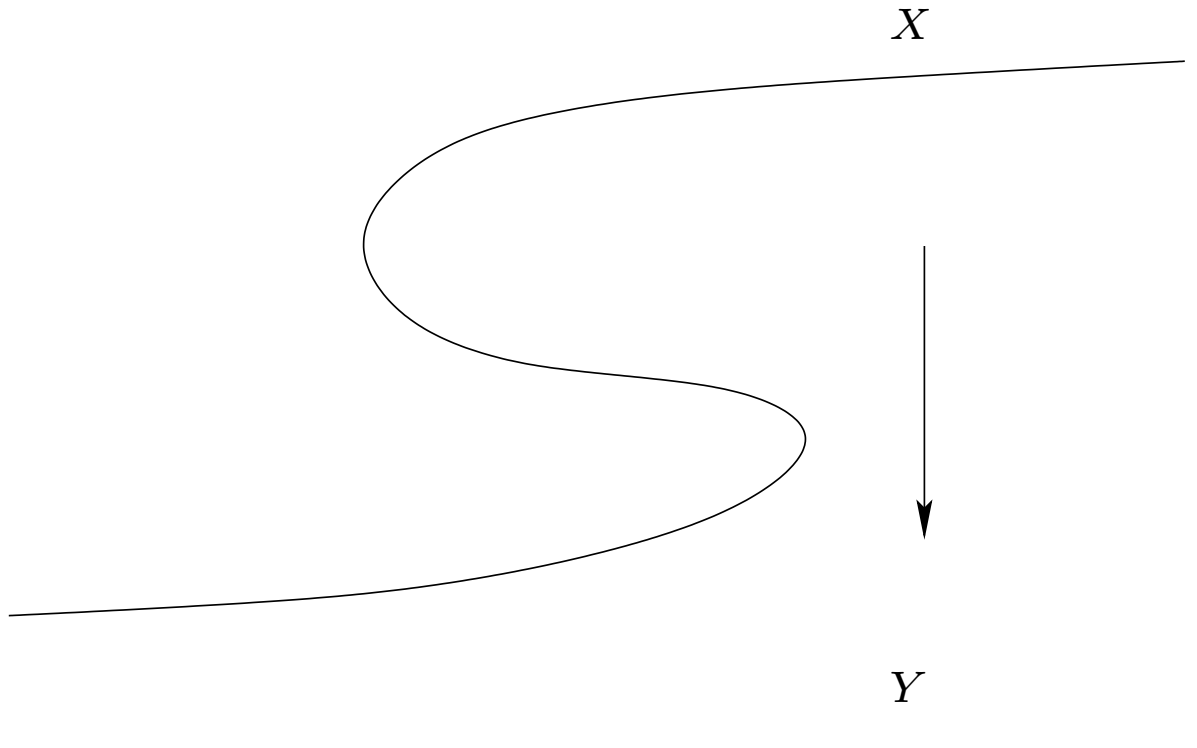
The vertical projection on this picture has degree ± 1.

b) If X and Y are not orientable, let $\widetilde{X} \to X$ and $\widetilde{Y} \to Y$ be orientable coverings of degree 2. Their covering groups are $\{\pm 1\}$.

If there exists a lifting

$$\widetilde{f} : \widetilde{X} \to \widetilde{Y},$$

which commutes with the $\{\pm 1\}$ action, we define $\deg f = \deg \widetilde{f}$.

f is called *orientable* if $\widetilde{f}$ exists.

For connected X and Y and orientable f, $\deg f$ is defined as an integer, up to sign.

If all preimages of a regular value belong to one affine chart, the degree can be computed by the usual formula. Orientability of the map guarantees that the result does not depend on the chart.

Theorem 1 *The degree of the real Wronski map*

$$\phi : G_{\mathbf{R}}(m, m+p) \to \mathbf{RP}^{mp}$$

is $\pm I(m, p)$, where $I(m, p) \neq 0$ iff $m + p$ is odd. For odd $m + p$ we have $I(m, p) =$

$$\frac{1! \, 2! \cdots (p-1)!(pm/2)!}{(m-p+2)!(m-p+4)! \cdots (m+p-2)!}$$

$$\times \frac{(m-1)!(m-2)! \cdots (m-p+1)!}{\left(\frac{m-p+1}{2}\right)! \left(\frac{m-p+3}{2}\right)! \cdots \left(\frac{m+p-1}{2}\right)!},$$

Some values of $I(m, p)$:

$m =$	3	4	5	6	7	8	9
$p = 2$	1	0	2	0	5	0	14
$p = 3$	0	2	0	12	0	110	0
$p = 4$		0	12	0	286	0	12376
$p = 5$			0	286	0	33592	0
$p = 6$				0	33592	0	...

Notice: $I(2k+1, 2)$ is the k-th Catalan number.

Corollary 1 *For odd $m + p$, the preimage of every point in $\mathbf{RP}^{mp}$ under the Wronski map is non-empty. For a generic point it contains at least $I(m, p)$ points.*

Corollary 2 *If the given subspaces in the Schubert problem are osculating the normal rational curve at real points, and $m+p$ is odd, then the problem has at least $I(m, p)$ real solutions.*

Corollary 3 *If $d+n$ is odd, there are maximally inflected curves of degree d in $\mathbf{P}^n$ with arbitrary prescribed real inflection points. There are at least $I(d-n, n+1)$ such non-equivalent curves.*

For $p = 2$ and odd m, the lower estimate $I(m, 2)$ in all these Corollaries 1-3 is best possible:

Example 1 *If $p = 2$, the preimage in $G_{\mathbf{R}}(m, m+2)$ of some points in $\mathbf{RP}^{2m}$ under the real Wronski map consists of $I(m, 2)$ simple points.*

Theorem 2 *If m and p are both even, then the Wronski map omits a non-empty open set in $\mathbf{RP}^{mp}$.*

Corollary 4 *If m and p are both even, there is a non-empty open set of pole configurations for which the pole placement problem has no real solutions.*

Question Is the real Wronski map surjective when both m and p are odd?

7. Combinatorial interpretation of $I(m, p)$.

For any initial segment, a has at least as many votes as b, b has at least as many as c, etc., but the election ends with a tie between all candidates.
For example,

$$\sigma = (a\ a\ b\ c\ b\ a\ c\ b\ c).$$

The number of ballot sequences with p candidates and mp voters equals $d(m, p)$, the degree of the complex Grassmann variety (Frobenius, MacMahon).

An *inversion* is a pair of ballots where the order of candidates is non-alphabetical. For the sequence above, the total number of inversions $\text{inv}\,\sigma = 6$.

Then:

$$I(m, p) = \sum_{\text{all ballot seq}} (-1)^{\text{inv}\,\sigma}.$$

Theorem (Dennis White, 2000).
$I(m, p) = 0$ if $m + p$ is even, and for odd $m+p$, $I(m,p) =$ the number of shifted standard Young tableaux with $(m+p-1)/2$ cells in the top row, $(m-p+1)/2$ cells in the bottom row, and of height p:

1	2	4	5
	3	6	8
		7	9

Explicit expression for the number of such tableaux was given by Thrall in 1952 ("Hook Formula").

Boris and Michael Shapiro Conjecture: *If all zeros of the Wronskian determinant of several polynomials are all real, the polynomials can be made real by a non-degenerate linear transformation.*

The following theorem proves the conjecture for the case of two polynomials.

Connection with rational functions: put $f = f_1/f_2$. Then critical points of f in $\mathbf{C}$ coincide with zeros of $W(f_1, f_2)$.

Theorem 3 *If all critical points of a rational function belong to a circle (on the Riemann sphere), it maps this circle into a circle.*

Proof of Theorem 3.

Equivalence of rational functions: $f \sim g$ if $f = \ell \circ g$, $\deg \ell = 1$.

Theorem (L. Goldberg, 1990) *Given $2m$ points on the Riemann sphere, there exist $d(m,2)$ classes of rational functions with these critical points.*

Theorem 4 *Given $2m$ points on a circle, there exist $d(m,2)$ classes of rational functions with these critical points, mapping this circle into itself.*

Theorem 3 follows.

Sketch of the proof of Theorem 4.

R is the class of rational functions f such that:
$f(\mathbf{T}) \subset \mathbf{T}$, $\deg f = m+1$, all critical points are
simple and belong to $\mathbf{T}$, $f(1) = 1$, $f'(1) = 0$.

Net $\gamma = f^{-1}(\mathbf{T})$, modulo orientation-preserving
homeomorphisms of $\overline{\mathbf{C}}$, fixing 1, symmetric with
respect to $\mathbf{T}$.
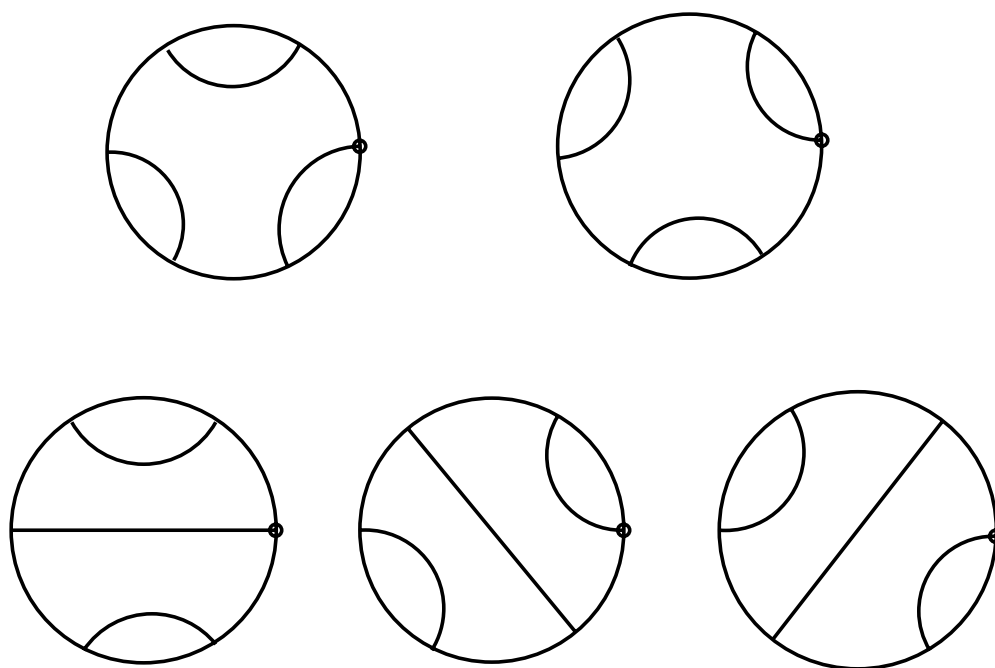
All nets for $d = 4$ (parts inside $\mathbf{T}$).

Lemma. *There are $d(m, 2)$ nets.*

A *labeling* of a net: non-negative function on the set of edges, symmetric with respect to **T**, and satisfying

$$\sum_{e \in \partial G} p(e) = 2\pi \quad \text{for every face} \quad G.$$
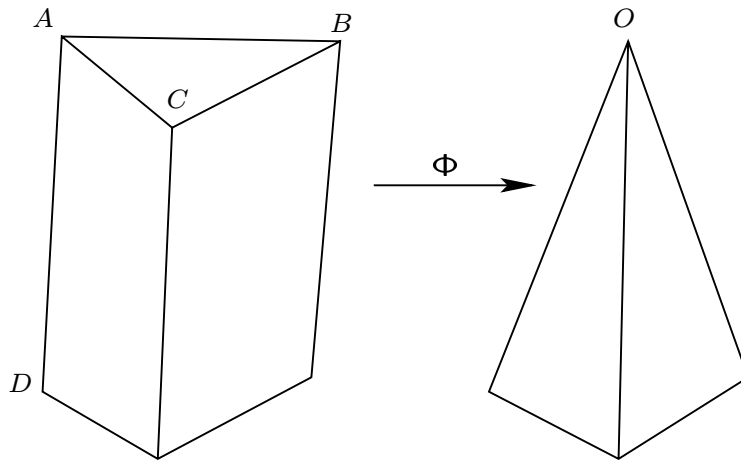
For example: $p(e) = \text{length } f(e)$.

critical set: $2m$ points on **T**, including $1 \in \mathbf{T}$.

Fix a net γ. Let L_γ be the space of all labelings and Σ_γ the space of all critical sets. They are convex polytopes of the same dimension. The Uniformization Theorem gives

$$\Phi_\gamma : L_\gamma \to \Sigma_\gamma.$$

This map is continuous for each γ.

Proof of surjectivity of Φ.
a) Extension to $\overline{L}_\gamma \to \overline{\Sigma}_\gamma$.
b) Combinatorics of the boundary map (degeneracy of rational functions)
c) Topological lemma: *Let Φ be a continuous map of closed convex polytopes of the same dimension. If the preimage of every closed face (of any dimension) has homology groups of one point (in particular, this preimage is non-empty and connected), then Φ is surjective.*

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