On the spherical derivative of a rational function

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Abstract

For a rational function f we consider the norm of the derivative with respect to the spherical metric and denote by $K(f)$ the supremum of this norm. We give estimates of this quantity $K(f)$ both for an individual function and for sequences of iterates.

Keywords: rational function, spherical derivative, characteristic exponent.

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A rational function is a holomorphic map from the Riemann sphere into itself. We equip the Riemann sphere with the usual spherical metric whose length and area elements are

$$ds = \frac{|dz|}{1+|z|^2} \quad \text{and} \quad dA = \frac{dxdy}{(1+|z|^2)^2}.$$

So the norm of the derivative with respect to the spherical metric is

$$\|f'\|(z) := |f'(z)|(1+|z|^2)/(1+|f(z)|^2).$$

In this paper we study the quantity $K(f) = \max_{\overline{\mathbf{C}}} \|f'\|$. d'Ambra and Gromov [1] proposed to study the rate of growth of $\sup \|(f^n)'\|$ as $n \to \infty$ for

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the iterates f^n of smooth maps of Riemannian manifolds, especially those maps in a given class for which this growth rate is the smallest possible. Such maps are called "slow". Slow maps of an interval and slow Hamiltonian diffeomorphisms of a 2-torus have been investigated in [3, 16] and [2].

Let f be a rational function of degree d. As the map $f : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ is d-to 1, we conclude that

$$\int\int_{\mathbf{C}} \|f'\|^2 dA = d(f) \int\int_{\mathbf{C}} dA.$$

This implies that

$$K(f) \geq \sqrt{d(f)} \tag{1}$$

We ask how small can $K(f)$ be for a function of given degree.

It is known that $K(f) \geq 2$ for all rational functions of degree at least 2. In fact this holds for all smooth maps of the sphere into itself which satisfy $\deg(f) \notin \{0, 1, -1\}$ [10]. It is not known whether $K(f) = 2$ can hold for rational functions of degrees 3 or 4.

An interesting question is whether (1) is best possible in certain sense. We have

Theorem 1. *There exists an absolute constant C with the following property. For every $d \geq 2$ there exists a rational function of degree exactly d such that*

$$K(f) \leq C\sqrt{d}.$$

An analogous result was obtained by Gromov [10, Ch. 2D] for smooth maps of spheres of arbitrary dimension.

Littlewood [14] and Hayman [12] studied the quantity

$$\phi(d) = \sup_{\deg f = d} \sup_{R>0} \frac{1}{R} \int\int_{|z|\leq R} \frac{|f'(z)|}{1+|f(z)|^2} dxdy,$$

where the sup is taken over all rational functions of degree d. For polynomials $\phi(d)$ was also studied in [9, 6, 13]. It is easy to see that $\phi(d) \leq \pi\sqrt{d}$ and Hayman obtained $\phi(d) \geq c_1\sqrt{d}$ using rational approximations of elliptic functions. Our Theorem 1 implies this with a more elementary proof. Indeed, by a change of the independent variable,

$$\phi(d) = \sup_{\deg f = d} \int\int_{|z|\leq 1} \|f'\| \frac{dxdy}{1+|z|^2}.$$

Denote

$$f^{\#} = |f'|/(1+|f|^2). \tag{2}$$

Let f be the function from Theorem 1. By rotating the sphere of the independent variable we may achieve that

$$\int\int_{|z|\leq 1} (f^{\#})^2 dxdy \geq \pi d/2,$$

because the spherical area of the image sphere is π and it is covered d times. Let $M = \max_{|z|\leq 1} f^{\#}$. Theorem 1 implies that $M \leq C\sqrt{d}$, so

$$\begin{aligned}\phi(d) &\geq \int\int_{|z|\leq 1} f^{\#} dxdy \geq M^{-1} \int\int_{|z|\leq 1} (f^{\#})^2 dxdy \\ &\geq C^{-1} d^{-1/2} \pi d/2 = \pi C\sqrt{d}/2.\end{aligned}$$

Proof of Theorem 1. For every positive integer n, consider the function

$$f_n(z) = \prod_{k=-n}^{n} \tanh(z+2k).$$

We first show that

$$\frac{|f_n|}{1+|f_n|^2} \leq C, \tag{3}$$

where C is independent of n. We have

$$||\tanh z| - 1| \leq 4e^{-2x} < 1, \quad x = |\Re z| \geq 1. \tag{4}$$

Now fix $z \in \mathbf{C}$ and let m be an integer such that $|2m - \Re z| \leq 1$. For $|w-z| \leq 1/2$ put

$$g(w) = \begin{cases} \tanh(w-2m), & \text{if} \quad |m| \leq n, \\ 1, & \text{otherwise} \end{cases}$$

Then evidently

$$\frac{|g'(w)|}{1+|g(w)|^2} \leq \frac{|\tanh'(w)|}{1+|\tanh(w)|^2} \leq C_0. \tag{5}$$

We write $f_n = gh$, $f_n' = g'h + gh'$, and estimate h using (4):

$$|h(w)| \leq \prod_{k=0}^{\infty} \left(1 + 4e^{-2(2k+1/2)}\right)^2 =: C_1.$$

Now h is holomorphic in $|w-z| \leq 1/2$, so by Cauchy's theorem,

$$|h'(z)| \leq 4C_1.$$

Next we estimate $h(z)$ from below using (4) again:

$$|h(z)| \geq \prod_{k=0}^{\infty} \left(1 - 4e^{-2(2k+1)}\right)^2 =: C_2.$$

Combining all these estimates we obtain

$$\frac{|f_n'(z)|}{1+|f_n(z)|^2} \leq \frac{C_1|g'(z)| + 4C_1|g(z)|}{1+C_2^2|g(z)|^2} \leq C_2^{-2}(C_1C_0 + 4C_1).$$

This proves (3)

When $|\Re z| \geq n+1$, we will obtain better estimates. Using

$$|\tanh'(z)| = |\cosh(z)^2| \leq 16e^{-2|x|}, \quad |x| = |\Re z| \geq 1,$$

we obtain for $z > 2n$ and $\xi = \Re z - 2n$:

$$|f_n'(z-2n)| \leq C_1 \sum_{k=0}^{\infty} |\cosh(\xi+2k)|^2 \leq 16C_1 \sum_{k=0} e^{-2(\xi+2k)} = 16C_1C_3e^{-2\xi},$$

which gives

$$\frac{|_n'(z)|}{1+|f_n(z)|^2} \leq Ce^{-2(|x|-n)} \leq Ce^{-2|x|/n}$$

if $|x| \geq n+2$. Combining this with (3) we obtain

$$\frac{|f_n'(z)|}{1+|f_n(z)|^2} \leq Ce^{-2|x|/n}$$

for all z. As f_n has period π, there exists a rational function R_n such that $f_n(nz) = R_n(e^{2z})$. This rational function has degree $2n^2$ and the derivative satisfies $\|R_n'\| \leq Cn$. This completes the proof of Theorem 1.

Theorem 2. *There exists an absolute constant $c > 1$ with the property that*

$$K(f) \geq c\sqrt{d}$$

for all rational functions of degree $d \geq 2$.

This can be considered as an analog of a result of Tsukamoto [18]. He studied spherical derivatives (2) of meromorphic functions $F:\Delta\to\overline{\mathbf{C}}$, where Δ is the unit disc with the Euclidean metric and proved that there exists an absolute constant $c_1<1$ with the property that $\omega(F(\Delta))\leq c_1\pi$ for all meromorphic functions F satisfying $F^{\#}\leq 1$, where

$$\omega(F(\Delta))=\int_{|z|<1}(F^{\#}(z))^2dxdy.$$

We derive Theorem 2 from this result. In fact we show that Theorem 2 holds with $c=1/\sqrt{c_1}$.

Proof of Theorem 2. Proving by contradiction, we suppose that there exists a sequence f_m of rational functions of degrees m, such that

$$K(f_m)/\sqrt{m}\to b<1/\sqrt{c_1}. \tag{6}$$

Let ω be the spherical area measure, so that

$$\int_{\overline{\mathbf{C}}}d\omega=\pi,$$

and $\omega_m=f_m^*\omega$ the pull back of ω by f_m. Then

$$\int_{\overline{\mathbf{C}}}d\omega_m=\pi m. \tag{7}$$

It is easy to see that we can find discs $D_m=D(a_m,r_m)\subset\overline{\mathbf{C}}$ (with respect to the spherical metric) of radii r_m such that

$$\int_{D_m}d\omega=\pi/(mb^2)\quad\text{and}\quad\int_{D_m}d\omega_m\geq\pi/b^2. \tag{8}$$

To show this, choose r_m so that the first equation is satisfied and then integrate

$$F(a):=\int_{D(a,r_m)}d\omega_m(a)$$

with respect to a. Evidently $r_m\sim 1/b\sqrt{\pi m}, m\to\infty$.

Let a_m' be the point diametrically opposite to a_m, and let $\phi_m:\mathbf{C}\to\overline{\mathbf{C}}\backslash\{a_m'\}$ be the conformal map (inverse to a stereographic projection) such that $\phi_m(0)=a,\phi_m(\Delta)=D_m$, then

$$\phi_m^{\#}(z)\leq r_m(1+o(1)),\ m\to\infty, \tag{9}$$

uniformly with respect to z. Then $F_m = f_n \circ \phi_m$ is a normal family. Let $F = \lim F_m$. From (6), (9) follows that then $F^\# \leq 1$, but the area of $F(\Delta) \geq \pi/b^2$ in view of (8), contradicting the result of Tsukamoto.

Theorems 1 and 2 have analogs for maps $\mathbf{P}^1 \to \mathbf{P}^n$ which are stated and proved in the same way as for $n = 1$, using he Fubini–Study metric for the norm of the derivative. Constants C and c will of course depend on n.

Now we consider dynamical questions. By f^n we denote the n-th iterate, and our standing assumption is that $d(f) \geq 2$. We define

$$k_\infty(f) = \lim_{n\to\infty} \frac{1}{n} \log K(f^n).$$

The limit always exists because the sequence $a_n = \log K(f^n)$ is subadditive, $a_{m+n} \leq a_m + a_n$ and for every such positive sequence the limit $\lim_{n\to\infty} a_n/n$ exists and is equal to $\inf_n a_n/n$ (see, for example, Lemma 1.16 in [4]).

It follows that $k_\infty(f^m) = mk_\infty(f)$.

Notice that k_∞ is independent of the choice of a smooth Riemannian metric on the sphere, and is invariant under conjugation by conformal automorphisms. Obviously, $k_\infty(f) \leq \log K(f)$.

What is the smallest value of $k_\infty(f)$ for rational functions of given degree?

The trivial lower estimate of $K(f)$ gives

$$k_\infty(f) \geq (1/2) \log d(f). \tag{10}$$

We will see that equality never happens, and that the Lattés functions are not extremal for minimizing k_∞. For functions f_d of degree d from Theorem 1 we have

$$k_\infty(f_d)/\log d \to 1/2, \quad d \to \infty,$$

so the 1/2 in (10) cannot be replaced with a larger constant.

In [7] these quantities were studied for polynomials, in particular, the inequality $k_\infty(f) \geq \log d(f)$ was established for polynomials, with equality only if f is conjugate to z^d.

Let us consider a slightly different quantity, the *maximum characteristic exponent*

$$\chi_m(f) = \sup_z \limsup_{n\to\infty} \frac{1}{n} \log \|(f^n)'(z)\| \leq k_\infty(f). \tag{11}$$

The difference in the definitions of k_∞ and χ_m is in the order of $\max_z$ and $\lim_{n\to\infty}$. Przytycki proved in 1998 (reproduced in [17]) that the same quantity $\chi_m(f)$ can be obtained by taking the sup over periodic pints z of f, in which case the lim sup in (11) can be of course replaced by the ordinary limit. Moreover, he proved the following:

Theorem P. *For every $\epsilon > 0$ there exists a periodic point z such that*

$$\frac{1}{m}\log\|(f^m)'(z)\| \geq k_\infty - \epsilon, \tag{12}$$

where m is a period of z.

In particular, $k_\infty = \chi_m$, and one can replace $\sup_{z\in\overline{\mathbf{C}}}$ in (11) by sup *over all periodic points.*

Let $\chi_a(f)$ be the average value of

$$\log\|(f^n)'(z)\|$$

over the measure μ of maximal entropy. According to the multiplicative ergodic theorem, we have

$$\chi_a(f) = \lim_{n\to\infty}\frac{1}{n}\log\|(f^n)'(z)\|$$

almost everywhere with respect to μ.

Theorem 3. $\chi_a(f) \geq (1/2)\log d$, *this is best possible and equality holds only for Lattés examples.*

Proof. The estimate follows from the formula

$$\chi_a(f) = h(f)/\dim\mu,$$

where $h(f) = \log d$ is the topological entropy, and $\dim\mu$ is the Hausdorff dimension of the maximal measure, see, for example [8] and references therein. Obviously $\dim\mu \leq 2$ so we obtain our inequality. On the other hand, a theorem of A. Zdunik [20] says that $\dim\mu = 2$ can happen only for Lattés examples. This completes the proof of the theorem.

Now we consider the Lattés functions [15].

Proposition 1. *If L is a Lattés function then $\chi_m(L) \geq \log d(L)$.*

Proof. A Lattés function can be defined by a functional equation

$$F(\lambda z) = L \circ F(z), \quad d(L) = \lambda^2,$$

where F is an elliptic function with a critical point at 0. Assuming without loss of generality that $F(0) = 0$ we conclude that 0 is a fixed point of L. Now the derivative at this fixed point is

$$\lambda \lim_{z \to 0} F'(\lambda z)/F'(z) = \lambda^2 = d(L),$$

from which follows that $\chi_m(L) \geq d(L)$.

To summarize, we have 4 quantities satisfying inequalities

$$\frac{1}{2} \log d \leq \chi_a(f) \leq \chi_m(f) = k_\infty(f) \leq \log K(f). \tag{13}$$

For Lattés functions we have

$$(1/2) \log d = \chi_a(L) < \chi_m(L) = \log d.$$

In the first inequality equality holds only for Lattés functions, but the second inequality is strict for them. We conclude that for all rational functions

$$k_\infty(f) = \chi_m(f) > (1/2) \log d.$$

On the other hand, for functions f_d constructed in Theorem 1, we have $\log K(f_d) \leq (1/2) \log d + O(1),\ d \to \infty$.

Which of the rest of inequalities (13) are strict and what are the conditions of equality? According to a private communication of Przytycki, the method of Zdunik can be used to show that $\chi_a(f) = \chi_m(f)$ is only possible when f is conjugate to $z^{\pm d}$ in which case both quantities are equal to $\log d$.

We conclude with the investigation of the equalty $K(f) = k_\infty(f)$.

Proposition. *Let* $M = \{z : \|f'(z)\| = K(f)\}$. *Then* $k_\infty = \log K(f)$ *if and only if* M *contains a cycle.*

Proof. If z is a point whose trajectory is in M, then $\|(f^m)'(z)\| = \|f'(z)\|^m$ so $k_\infty(f) = \log K(f)$.

Suppose now that $k_\infty = \log K(f)$. We claim that

$$\bigcap_{j=0}^{\infty} f^j(M) \neq \emptyset.$$

Indeed, otherwise for some m we have

$$\bigcap_{j=0}^{m} f^j(M) = \emptyset,$$

and if this holds, then

$$\begin{aligned} mk_\infty(f) &= k_\infty(f^m) \le \log K(f^m) = \log \max_z \prod_{j=0}^{m-1} \|f'(f^j(z)\| \\ &< m \log K(f), \end{aligned}$$

contrary to our assumption. This proves the claim.

The set M_∞ is forward invariant. All sets

$$\bigcap_{j=0}^{m} f^j(M)$$

are real algebraic subsets of $\overline{\mathbf{C}}$, and thus the set M_∞ is also real algebraic. If M_∞ is finite it must contain a cycle.

Suppose that M is infinite. Let $M_1, ...M_q$ be the list of connected components of M. Then there must be a cycle of components. Let $M_1, \ldots, M_k$ be this cycle, so that $f(M_j) \subset M_{j+1}$, $f(M_k) = M_1$. If any of these components contains singular points, then all singular points in these components form a finite invariant set, and this set must contain a cycle. If all M_j, $1 \le j \le k$ are smooth, then $f^k : M_1 \to M_1$ is an expanding map of a circle, so it must have a fixed point. Again we obtain a cycle in M. This completes the proof.

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