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A COUNTEREXAMPLE TO CARTAN'S CONJECTURE ON HOLOMORPHIC CURVES OMITTING HYPERPLANES

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ABSTRACT. In his 1928 thesis H. Cartan proved a theorem which can be considered as an extension of Montel's normality criterion to holomorphic curves in complex projective plane $\mathbf{P}^2$. He also conjectured that a similar result is true for holomorphic curves in $\mathbf{P}^n$ for any n. A counterexample to this conjecture is constructed for any $n \geq 3$.

The following theorem of Borel may be considered as an extension of Picard's theorem to holomorphic mappings of the complex plane $\mathbf{C}$ to complex projective space.

Borel's Theorem. *Let $f_1, \dots, f_p$ be a system of entire functions without zeros and*

$$f_1 + \dots + f_p = 0. \tag{1}$$

Then the set of indices $\{1, \dots, p\}$ can be partitioned into disjoint subsets $\{I\}$ such that $|I| \geq 2$, and for every I the functions f_j, $j \in I$, are proportional and their sum is zero.

According to the so-called Bloch principle, to every theorem of Picard type should correspond a Montel-type theorem for families of functions in the unit disk. The following statement is known as

Cartan's Conjecture ([2, 3]). *Let $\mathcal{F}$ be an infinite family of p-tuples of holomorphic functions $f = (f_1, \dots, f_p)$ without zeros in the unit disk $\mathbf{U}$ satisfying the Borel equation* (1).

Then there exists an infinite subsequence $\mathcal{L}$ having the following property.

There exists a partition of indices $P = \{1, \dots, p\}$ into disjoint sets $\{S\}$ and each S contains a subset I with at least two elements, which may be equal to S itself. These satisfy the following properties for $f \in \mathcal{L}$:

(i) For each S and $j, k \in I \subset S$ the sequence $\{f_j/f_k\}$ is convergent (uniformly on compacta, to a non-zero function).

(ii) If $j \in S \backslash I$ and $k \in I \subset S$ then f_j/f_k converges to 0.

(iii) Given $k \in I \subset S$,

$$\sum_{j \in I} f_j/f_k \text{ converges to } 0.$$

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When $p = 3$ the statement is (almost) equivalent to the Montel theorem, which asserts that a family of meromorphic functions in the unit disk omitting three given values is normal. Cartan [2], see also [3, Ch. VIII], proved a partial result:

Let $\mathcal{F}$ be as above. Then there exists a subsequence $\mathcal{L} \subset \mathcal{F}$ having one of the following properties:

(a) *The full set P of indices satisfies* (i), (ii) *and* (iii) *(with single set $S = P$), or*

(b) *There are two disjoint subsets S_1 and S_2 in P, each containing at least two elements, satisfying the three conditions* (i), (ii) *and* (iii).

The point is that S_1 and S_2 in (b) may not cover the whole set of indices P. This result implies that Cartan's conjecture is true for $p = 3$ and $p = 4$ [2]. We show that it fails for $p = 5$.

Example. It is convenient to work in the rectangle $R = \{x + iy : |x| < \pi, 0 < y < 1\}$ instead of the unit disk. For every natural integer $n > 12 > 4e$ consider the function $h(z) = h_n(z) = \exp(n \exp iz)$, $z \in R$. We have

$$\log |h_n(x + iy)| = n \cos x \exp(-y).$$

The set $\{z \in R : |h_n(z)| < 3\}$ consists of two components: left and right. We denote the right component by D_n so that as $n \to \infty$, $D_n \to R \cap \{x \geq \pi/2\}$. Choose a diffeomorphism p of the disk $\{w : |w| \leq 3\}$ onto itself with the following properties:

$$p(w) = w, \quad |w| = 3,$$

$$p(0) = 1$$

and

$$p \quad \text{is conformal} \quad \text{for} \quad |w| < 2.$$

Put

$$\tilde{G}_n(z) = \begin{cases} p \circ h_n(z), & z \in D_n, \\ h_n(z), & z \in R \backslash D_n. \end{cases}$$

Then we can find a diffeomorphism $\phi_n : R \to R$, continuous in $\bar{R}$ with

(2) $$\phi_n(0) = 0, \quad \phi_n(\pm\pi) = \pm\pi$$

such that

$$G_n = \tilde{G}_n \circ \phi_n^{-1}$$

is holomorphic in R. This ϕ_n is obtained by solving a Beltrami equation [1]

$$\frac{\partial \phi_n}{\partial \bar{z}} = \mu \frac{\partial \phi_n}{\partial z},$$

where μ is a smooth function, $|\mu(z)| \leq c \leq 1$, $z \in R$, c an absolute constant, and

(3) $$\operatorname{supp} \mu = K_n = \{z \in R : \Re z > 0, \ 2 \leq |h_n(z)| \leq 3\}.$$

We claim that

(4) $$\phi_n(z) - z \to 0, \quad n \to \infty$$

uniformly on R. Indeed, $\{\phi_n\}$ is a family of quasiconformal homeomorphisms of R with uniformly bounded dilatation, so this family is precompact (the topology of uniform convergence). Any limit function ϕ of the family is conformal everywhere in R except perhaps the segment

$$K = \{\pi/2 + it : 0 < t < 1\} = \lim_{n\to\infty} K_n.$$

But K is a removable singularity for homeomorphisms conformal in the complement of K. So ϕ is a conformal automorphism of R and (2) implies that $\phi = \mathrm{id}$. This proves (4). Notice that $G_n - 1$ has no zeros in $R \cap \{x > 0\}$ and G_n has no zeros in $R \cap \{x < 0\}$. It follows from (4) that

$$\log|G_n(x+iy) - 1| = (n + o(1)) \cos x \exp(-y), \quad x > 0 \tag{5}$$

and

$$\log|G_n(x+iy)| = (n + o(1)) \cos x \exp(-y), \quad x < 0, \tag{6}$$

when $n \to \infty$ uniformly on R. Now we define H_n by

$$G_n + H_n = 1. \tag{7}$$

Asymptotic equalities (5) and (6) imply respectively

$$\log|H_n(x+iy)| = (n + o(1)) \cos x \exp(-y), \quad x > 0 \tag{8}$$

and

$$\log|H_n(x+iy) - 1| = (n + o(1)) \cos x \exp(-y), \quad x < 0, \tag{9}$$

as $n \to \infty$ uniformly on R.

Now we set $a = \pi - 1/(e+1)$ and define

$$f_n^1(z) = \exp\{n(z+a)\},\ f_n^2(z) = \exp\{n(-z+a)\},$$

$$f_n^3 = G_n - f_n^1,\ f_n^4 = H_n - f_n^2,\ f_n^5(z) \equiv -1.$$

From this definition and (7) follows that (1) is satisfied. Furthermore we have in view of (5), (6), (8) and (9)

$$|G_n| < |f_n^1| \qquad \text{and} \qquad |H_n| < |f_n^2| \qquad \text{in} \quad R \tag{10}$$

for n large enough.

Inequalities (10) show that all five functions f^j are zero-free in R if n is large enough.

Now we show that the conclusion of Cartan's conjecture is not valid for the functions of our sequence. This is because f_n^5 cannot be in the same class S with any other function f_n^j, $1 \le j \le 4$. Indeed, when j is odd we have

$$\log|f_n^j(z)| = (n + o(1))(\Re z + a), \quad n \to \infty,$$

so

$$f_n^j\left(-\pi + \frac{1}{2(e+1)} + \frac{i}{2}\right) \to 0 \qquad \text{and} \qquad f_n^j(i/2) \to \infty, \quad n \to \infty.$$

A similar argument works for even j. In this case

$$f_n^j\left(\pi - \frac{1}{2(e+1)} + \frac{i}{2}\right) \to 0 \qquad \text{and} \qquad f_n^j(i/2) \to \infty, \quad n \to \infty.$$

So $f_n^5 \equiv -1$ cannot be included in any class S described in (i) and (ii) of Cartan's conjecture.

Remarks. The simplest counterexample for any $p > 6$ can be constructed by adding non-zero constant functions f_n^j with the properties

$$\sum_{j=6}^{p} f_n^j = 0$$

and $|f_n^j| = b^{-n}$, $6 \leq j \leq p$, where $1 < b < \exp\{1/(e+1)\}$. These new functions may be included in one class S with f_n^5 but then (iii) fails for this class. Our example for $p = 5$ shows that even a partition into classes S, $\operatorname{card} S \geq 2$, which satisfy (i) and (ii), is impossible. Examples with this property can also be constructed for any $p > 5$.

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