Exceptional values in holomorphic families of entire functions

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In 1926, Julia [5] studied singularities of implicit functions defined by equations $f(z, w) = 0$, where f is an entire function of two variables. Among other things, he investigated the exceptional set P consisting of those w for which such equation has no solutions z. In other words, P is the complement of the projection of the analytic set $\{(z, w) : f(z, w) = 0\}$ onto the second coordinate. Julia proved that P is closed and cannot contain a continuum, unless it coincides with w-plane. Lelong [6] and Tsuji [9], [10, Thm. VIII.37] independently improved this result by showing that the logarithmic capacity of P is zero if $P \neq \mathbf{C}$. In the opposite direction, Julia [5] proved that every discrete set $P \subset \mathbf{C}$ can occur as the exceptional set. He writes: *"Resterait à voir si cet ensemble, sans être continu, peut avoir la puissance du continu"*.[1]

According to Alan Sokal (private communication), the same question arises in holomorphic dynamics, when one tries to extend to holomorphic families of transcendental entire functions the results of Lyubich [8] on holomorphic families of rational functions.

In this paper, we show that in general, the result of Lelong and Tsuji is best possible: every closed set of zero capacity can occur as an exceptional set (Theorem 2). Then we study a related problem of dependence of Picard exceptional values of the function $z \mapsto f(z, w)$ on the parameter w (Theorem 3).

It is known that the exceptional set P is discrete in the important case that $z \mapsto f(z, w)$ are functions of finite order. This was discovered by Lelong

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[1]It remains to see whether this set, without being a continuum, can have the power of continuum.

in [6] and later the result was generalized to the case of multi-dimensional parameter w in [7, Thm. 3.44].

We also mention that the set P has to be analytic in certain holomorphic families of entire functions with finitely many singular values, considered in [1, 2]. These families may consist of functions of infinite order.

We begin with a simple proof of a version of Lelong's theorem on functions of finite order.

Theorem 1. *Let D be a complex manifold, and $f : \mathbf{C} \times D \to \mathbf{C}$ an analytic function, such that the entire functions $z \mapsto f(z, w)$ are not identically equal to zero and are of finite order for all $w \in D$. Then the set*

$$P = \{w \in D : \; (\forall z \in \mathbf{C}) \, f(z, w) \neq 0\} \tag{1}$$

is analytic.

Corollary. *Let D be a region in $\mathbf{C}$, and $f : \mathbf{C} \times D \to \mathbf{C}$ an analytic function, such that entire functions $z \mapsto f(z, w)$ are of finite order for all $w \in D$. Then the set P as in* (1) *is discrete or $P = D$.*

Indeed, the set $A = \{w \in D : f(., w) \equiv 0\}$ is discrete (unless $f = 0$ when there is nothing to prove). So there exists a function g holomorphic in D whose zero set is A, and such that f/g satisfies all conditions of Theorem 1. □

Remarks.

1. In general, when D is of dimension greater than one, and the set A is not empty, one can only prove that $A \cup P$ is contained in a proper analytic subset of D, unless $A \cup P = D$, [7, Thm 3.44].

2. If the order of $f(., w)$ is finite for all $w \in D$ then this order is bounded on compact subsets of D [7, Thm. 1.41].

Proof of Theorem 1. We assume without loss of generality that

$$f(0, w) \neq 0 \quad \text{for} \quad w \in D. \tag{2}$$

(shift the origin in $\mathbf{C}$ and shrink D, if necessary), and that the order of the function $f(., w)$ does not exceed λ for all $w \in D$ (see Remark 2 above).

Let p be an integer, $p > \lambda$. Then, for each w, f has the Weierstrass representation

$$f(z, w) = e^{c_0 + \ldots + c_p z^p} \prod_{a : f(a,w)=0} \left(1 - \frac{z}{a}\right) e^{z/a + \ldots + z^p/pa^p},$$

where a are the zeros of $f(.,w)$ repeated according to their multiplicities, and c_j and a depend on w. Taking the logarithmic derivative, differentiating it p times, and substituting $z=0$, we obtain for each $w \in D$:

$$F_p(w) = \frac{d^p}{dz^p}\left(\frac{df}{f\,dz}\right)\Big|_{z=0} = p! \sum_{a: f(a,w)=0} a^{-p-1}.$$

The series in the right hand side is absolutely convergent because of our choice of p. The functions F_p are holomorphic in D, in view of (2). Clearly $w \in P$ implies $F_p(w) = 0$ for all $p > \lambda$. In the opposite direction, $F_p(w) = 0$ for all $p > \lambda$ means that all but finitely many derivatives with respect to z at $z = 0$ of the unction df/fdz meromorphic in $\mathbf{C}$ are equal to zero, so this meromorphic function is a polynomial, and thus $f(z,w) = \exp(c_0 + \ldots + c_p z^p)$, that is $w \in P$. □

Our main result shows that the restriction of finiteness of order cannot be removed in Theorem 1. In general, if $D \subset \mathbf{C}$, one can only assert that P is a closed set of zero capacity, unless $P = D$. That the exceptional set P has zero capacity was proved in [6, 9].

Theorem 2. *Let $D \subset \mathbf{C}$ be the unit disc, and P an arbitrary compact subset of D of zero capacity. Then there exists a holomorphic function $f : \mathbf{C} \times D \to \mathbf{C}$, such that for every $w \in P$ the equation $f(z,w) = 0$ has no solutions, and for each $w \in D \backslash P$ it has infinitely many solutions.*

It is not clear whether a similar result holds with multidimensional parameter space D and arbitrary closed pluripolar set $P \subset D$.

Proof. Let $\phi : D \to D\backslash P$ be a universal covering. Let S be the set of singular points of ϕ on the unit circle. Then S is a closed set of zero Lebesgue measure.

(We recall a simple proof of this fact. As a bounded analytic function, ϕ has radial limits almost everywhere. It is easy to see that a point where the radial limit has absolute value 1 is not a singular point of ϕ. Thus the radial limits exist and belong to P almost everywhere on S. Let u be the "Evans potential" of P, that is a harmonic function in $D\backslash P$, continuous in $\overline{D}$ and such that $u(\zeta) = 0$ for $\zeta \in \partial D$ and $u(\zeta) = -\infty$ for $\zeta \in P$. Such function exists for every compact set P of zero capacity. Now $v = u \circ \phi$ is a negative harmonic function in the unit disc, whose radial limits on S are equal to $-\infty$, thus $|S| = 0$ by the classical uniqueness theorem.)

According to a theorem of Fatou, (see, for example, [3, Ch. VI]), for every set S of zero Lebesgue measure on ∂D, there exists a holomorphic function g in D, continuous in $\overline{D}$ and such that

$$\{\zeta \in \overline{D} : g(\zeta) = 0\} = S.$$

In particular, g has no zeros in D.

Now we define the following set $Q \subset \mathbf{C} \times D$:

$$Q = \{(1/g(\zeta), \phi(\zeta)) : \zeta \in D\}.$$

It is evident that the projection of Q on the second coordinate equals $D \backslash P$. It remains to prove that the set Q is analytic. For this, it is enough to establish that the map

$$\Phi : D \to \mathbf{C} \times D, \quad \Phi(\zeta) = (1/g(\zeta), \phi(\zeta))$$

is proper. Let $K \subset \mathbf{C} \times D$ be a compact subset. Then the closure of $\Phi^{-1}(K)$ in $\overline{D}$ is disjoint from S because g is continuous and $1/g(\zeta) \to \infty$ as $\zeta \to S$. On the other hand, for every point $\zeta \in \partial D \backslash S$, the limit

$$\lim_{\zeta \to \zeta_0} \phi(\zeta)$$

exists and has absolute value 1. So $\Phi^{-1}(K)$ is compact in D.

Now the existence of the required function f follows from the solvability of the Second Cousin problem [4, sect. 5.6]. □

Notice that the map Φ constructed in the proof is an immersion.

We recall that a point $a \in \mathbf{C}$ is called an exceptional value of an entire function f if the equation $f(z) = a$ has no solutions. Picard's Little theorem says that a non-constant entire function can have at most one exceptional value.

Let f be an entire function of z depending of the parameter w holomorphically, as in Theorems 1 and 2, and

assume in the rest of the paper that for all $w \in D$, $f(.,w) \neq$ const.

Let $n(w) \in \{0, 1\}$ be the number of exceptional values of $f(., w)$.

Question 1. What can be said about $n(w)$ as a function of w?

Example 1. $f(z,w) = e^z + wz$. We have $n(0) = 1$ and $n(w) = 0$ for $w \neq 0$.

Example 2. $f(z,w) = (e^{wz} - 1)/w$ for $w \neq 0$, and $f(z,0) = z$. We have $n(0) = 0$, while $n(w) = 1$ for $w \neq 0$. The exceptional value $a(w) = -1/w$ tends to infinity as $w \to 0$.

Thus n is neither upper nor lower semicontinuous.

Example 3. Let

$$f(z,w) = \int_{-\infty}^{z} (\zeta + w)e^{-\zeta^2/2} d\zeta,$$

where the contour of integration consists of the negative ray, passed left to right, followed by a curve from 0 to z. We have $f(z,0) = -e^{-z^2/2}$, which has exceptional value 0, so $n(0) = 1$. It is easy to see that there are no exceptional values for $w \neq 0$, so $n(w) = 0$ for $w \neq 0$. Thus $n(w)$ is the same as in Example 1, but this time we have an additional feature that the set of singular values of $f(.,w)$ is finite for all $w \in \mathbf{C}$, namely, there is one critical value $f(-w,w)$ and two asymptotic values, 0 and $\sqrt{2\pi}w$.

Question 2. Suppose that $n(w) \equiv 1$, and let $a(w)$ be the exceptional value of $f(.,w)$. What can be said about $a(w)$ as a function of w? Is it analytic?

The answer to the last question is positive for functions f of finite order in z. To show this, we first prove the following weak continuity property of the set of exceptional values which is true for all *meromorphic functions* holomorphically depending on parameter. We always assume that $z \mapsto f(z,w)$ is non-constant for all w. Denote

$$A(w) = \{a \in \overline{\mathbf{C}} : \ (\forall z \in \mathbf{C})\, f(z,w) \neq a\}.$$

Proposition 1. *For every $w_0 \in D$ and every $\epsilon > 0$ there exists $\delta > 0$ such that $|w - w_0| < \delta$ implies that $A(w)$ is contained in the ϵ-neighborhood of $A(w_0)$ with respect to the spherical metric.*

Proof. Let U be the open ϵ-neighborhood of $A(w_0)$. Then $K = \overline{\mathbf{C}} \backslash U$ is compact, so there exists $r > 0$ such that the image of the disc $|z| < r$ under $f(.,w_0)$ contains K. Then by Hurwitz's theorem, for every w close enough to w_0 the image of the disc $|z| < r$ under $f(.,w)$ will also contain K. □

Example 4. Let $f(z,w) = (e^{we^z} - 1)/w, \ w \neq 0$ and $f(z,0) = e^z$. Then f is an entire function of two variables and $n(w) \equiv 1$. However $a(w) =$

$-1/w,\ w \neq 0$ and $a(0) = 0$, so a is a discontinuous function of w.

We could not find an example with $n(w) \equiv 1$ and non-isolated points of discontinuity of $a(w)$. It is also unclear whether $a(w)$ can be continuous but not analytic.

Corollary 1. *The set of meromorphic functions having no exceptional values on the Riemann sphere is open.*

The set of entire functions whose only exceptional value is ∞ is not open as Example 2 above shows.

Corollary 2. *Suppose that $f(.,w)$ is entire and has an exceptional value $a(w) \in \mathbf{C}$ for all w. If $a(w)$ is bounded then it is continuous.*

Theorem 3. *Let D be a region in a complex manifold, and $f : \mathbf{C} \times D \to \mathbf{C}$ an analytic function, such that the entire functions $z \mapsto f(z,w)$ are non-constant and of finite order at most λ for all $w \in D$. Suppose that for each $w \in D$, the entire function $f(.,w)$ has an exceptional value $a(w) \in \mathbf{C}$. Then $w \mapsto a(w)$ is an analytic function in D.*

Example 4 shows that one cannot drop the assumption about finite order in this theorem.

Proof of Theorem 3. The assumptions of the theorem imply that

$$\begin{aligned} f(z,w) &= e^{g(z,w)} + a(w) \\ &= \exp(c_0(w) + c_1(w)z + \ldots + c_d z^d) + a(w) \\ &= a(w) + a_0(w) + a_1(w)z + a_2(w)z^2 + \ldots, \end{aligned}$$

where $a + a_0$ as well as a_j with $j \geq 1$ are analytic functions of w. We want to prove that a is analytic in w. We will assume without loss of generality that

$$a_1 \not\equiv 0 \tag{3}$$

This can be achieved by a shift of the origin in z-plane.

Consider the formulas that express the a_j in terms of c_j:

$$\begin{array}{rcl} a_0 & = & \exp c_0, \\ a_1 & = & c_1 \exp c_0, \\ a_2 & = & (c_2 + c_1^2/2) \exp c_0, \\ a_3 & = & (c_3 + c_1 c_2 + c_1^3/6) \exp c_0, \\ \ldots & & , \\ a_n & = & R_n(c_1, \ldots, c_n) \exp c_0, \\ \ldots & & . \end{array} \tag{4}$$

Here R_n is a polynomial which is of the form $c_n + R_n^*(c_1, ..., c_{n-1})$, where R_n^* is a polynomial without terms of first degree. Polynomials R_n are obtained by the recurrent algorithm which can be described as follows: Let $g = c_0 + c_1 z + \ldots + c_d z^d$, considered as a polynomial of $d+1$ variables $c_1, \ldots, c_d, z$, and put $g_0 = 1,\ g_n = g'_{n-1} + g_{n-1} g'$ for $n = 1, 2, \ldots$, where the prime indicates differentiation with respect to z. Then $R_n = g_n|_{z=0}$ as polynomials with respect to the variables c_j.

Now we consider the system of $d+1$ equations with respect to $c_0, \ldots, c_d$:

$$R_n(c_1, \ldots, c_n) = a_n h, \quad \text{for} \quad n = 1, \ldots, d+1,$$

where $h = \exp(-c_0)$. We eliminate the variables $c_1, \ldots, c_d$, expressing them consecutively from equations No 1 to d and substituting these expressions to all subsequent equations. The result will be one equation with respect to h. This is an algebraic equation of degree $d+1$, and it is easy to see by induction that the term of the highest degree in this equation is

$$\pm a_1^{d+1} h^{d+1}/d. \tag{5}$$

(Set $a_2 = a_3 = \ldots = 0$ and find c_d to see this).

In view of (3), $a_1 \neq 0$. So $h(w)$ is a solution of a non-trivial algebraic equation whose coefficients are analytic in D and the top degree term has the form (5). So h is bounded in a neighborhood U of any point w_0 such that $a_1(w_0) \neq 0$. Using Corollary 2 above we conclude that the function

$$a(w) = f(0, w) - h(w)$$

is continuous in U. So h is also continuous in U, and thus h is analytic, as a continuous solution of an algebraic equation with analytic coefficients. So a is also analytic in U. From the theorem on removable singularities we conclude that h and a may have only poles at the points w_0 such that $a_1(w_0) = 0$. Let us show that poles are impossible.

If $a(w)$ has a pole of order m at $w = 0$, then the assumptions of our theorem imply that

$$f(z, 0) = \lim_{w \to 0} e^{g(z,w)} + a(w), \tag{6}$$

where $a(w) = w^{-m}a^*(w)$, where a^* is a holomorphic function, $a^*(0) \neq 0$. Putting $z = 0$ in (6) we obtain $e^{g(0,w)} = e^{c_0(w)} = b(w)w^{-m}$, where b is holomorphic at 0 and $b(0) = -a^*(0)$. In view of (4), all c_j are holomorphic at 0, and $c_j(w) = O(w^m)$ as $w \to 0$, for every $j \geq 1$. Writing $c_j(w) = w^m c_j^*(w)$ and

$$g^*(z, w) = w^{-m}(g(z, w) - g(0, w)) = c_1^*(w)z + \ldots + c_d^*(w)z^d,$$

we obtain

$$f(z, 0) = \lim_{w \to 0} w^{-m} \left(b(w) \exp(w^m g^*(z, w)) + a^*(w) \right),$$

and so we conclude that $f(z, 0) = g^*(z, 0)$ is a polynomial, contradicting our assumption that $n(0) = 1$. □

It is plausible that a version of Theorem 3 holds for functions of infinite order: the only singularities of $a(w)$ are of the type described in Example 4. More precisely, if D is of dimension 1, and $n(w) \equiv 1$, the singularities of $a(w)$ form an discrete set S, $a(w) \to \infty$ as $w \to S$, but $a(w) \in \mathbf{C}$ for $w \in S$.

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