

# Exceptional values in holomorphic families of entire functions

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In 1926, Julia [5] studied singularities of implicit functions defined by equations  $f(z, w) = 0$ , where  $f$  is an entire function of two variables. Among other things, he investigated the exceptional set  $P$  consisting of those  $w$  for which such equation has no solutions  $z$ . In other words,  $P$  is the complement of the projection of the analytic set  $\{(z, w) : f(z, w) = 0\}$  onto the second coordinate. Julia proved that  $P$  is closed and cannot contain a continuum, unless it coincides with  $w$ -plane. Lelong [6] and Tsuji [12], [13, Thm. VIII.37] independently improved this result by showing that the logarithmic capacity of  $P$  is zero if  $P \neq \mathbf{C}$ . In the opposite direction, Julia [5] proved that every discrete set  $P \subset \mathbf{C}$  can occur as the exceptional set. He writes: “*Resterait à voir si cet ensemble, sans être continu, peut avoir la puissance du continu*”.<sup>1</sup>

According to Alan Sokal (private communication), the same question arises in holomorphic dynamics, when one tries to extend to holomorphic families of transcendental entire functions the results of Lyubich [8] on holomorphic families of rational functions.

In this paper, we show that in general, the result of Lelong and Tsuji is best possible: every closed set of zero capacity can occur as an exceptional set (Theorem 2). Then we study a related problem of dependence of Picard exceptional values of the function  $z \mapsto f(z, w)$  on the parameter  $w$  (Theorem 3).

It is known that the exceptional set  $P$  is discrete in the important case that  $z \mapsto f(z, w)$  are functions of finite order. This was discovered by Lelong

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<sup>1</sup>It remains to see whether this set, without being a continuum, can have the power of continuum.

in [6] and later the result was generalized to the case of multi-dimensional parameter  $w$  in [7, Thm. 3.44].

We also mention that the set  $P$  has to be analytic in certain holomorphic families of entire functions with finitely many singular values, considered in [1, 2]. These families may consist of functions of infinite order.

We begin with a simple proof of a version of Lelong's theorem on functions of finite order.

**Theorem 1.** *Let  $D$  be a complex manifold, and  $f : \mathbf{C} \times D \rightarrow \mathbf{C}$  an analytic function, such that the entire functions  $z \mapsto f(z, w)$  are not identically equal to zero and are of finite order for all  $w \in D$ . Then the set*

$$P = \{w \in D : (\forall z \in \mathbf{C}) f(z, w) \neq 0\} \quad (1)$$

*is analytic.*

**Corollary.** *Let  $D$  be a region in  $\mathbf{C}$ , and  $f : \mathbf{C} \times D \rightarrow \mathbf{C}$  an analytic function, such that entire functions  $z \mapsto f(z, w)$  are of finite order for all  $w \in D$ . Then the set  $P$  as in (1) is discrete or  $P = D$ .*

Indeed, the set  $A = \{w \in D : f(., w) \equiv 0\}$  is discrete (unless  $f = 0$  when there is nothing to prove). So there exists a function  $g$  holomorphic in  $D$  whose zero set is  $A$ , and such that  $f/g$  satisfies all conditions of Theorem 1.  $\square$

*Remarks.*

1. In general, when  $D$  is of dimension greater than one, and the set  $A$  is not empty, one can only prove that  $A \cup P$  is contained in a proper analytic subset of  $D$ , unless  $A \cup P = D$ , [7, Thm 3.44].

2. If the order of  $f(., w)$  is finite for all  $w \in D$  then this order is bounded on compact subsets of  $D$  [7, Thm. 1.41].

*Proof of Theorem 1.* We assume without loss of generality that

$$f(0, w) \neq 0 \quad \text{for } w \in D. \quad (2)$$

(shift the origin in  $\mathbf{C}$  and shrink  $D$ , if necessary), and that the order of the function  $f(., w)$  does not exceed  $\lambda$  for all  $w \in D$  (see Remark 2 above).

Let  $p$  be an integer,  $p > \lambda$ . Then, for each  $w$ ,  $f$  has the Weierstrass representation

$$f(z, w) = e^{c_0 + \dots + c_p z^p} \prod_{a: f(a, w)=0} \left(1 - \frac{z}{a}\right) e^{z/a + \dots + z^p/pa^p},$$

where  $a$  are the zeros of  $f(., w)$  repeated according to their multiplicities, and  $c_j$  and  $a$  depend on  $w$ . Taking the logarithmic derivative, differentiating it  $p$  times, and substituting  $z = 0$ , we obtain for each  $w \in D$ :

$$F_p(w) = \frac{d^p}{dz^p} \left( \frac{df}{f dz} \right) \Big|_{z=0} = p! \sum_{a: f(a,w)=0} a^{-p-1}.$$

The series in the right hand side is absolutely convergent because of our choice of  $p$ . The functions  $F_p$  are holomorphic in  $D$ , in view of (2). Clearly  $w \in P$  implies  $F_p(w) = 0$  for all  $p > \lambda$ . In the opposite direction,  $F_p(w) = 0$  for all  $p > \lambda$  means that all but finitely many derivatives with respect to  $z$  at  $z = 0$  of the function  $df/f dz$  meromorphic in  $\mathbf{C}$  are equal to zero, so this meromorphic function is a polynomial, and thus  $f(z, w) = \exp(c_0 + \dots + c_p z^p)$ , that is  $w \in P$ .  $\square$

Our main result shows that the restriction of finiteness of order cannot be removed in Theorem 1. In general, if  $D \subset \mathbf{C}$ , one can only assert that  $P$  is a closed set of zero capacity, unless  $P = D$ . That the exceptional set  $P$  has zero capacity was proved in [6, 12].

**Theorem 2.** *Let  $D \subset \mathbf{C}$  be the unit disc, and  $P$  an arbitrary compact subset of  $D$  of zero capacity. Then there exists a holomorphic function  $f : \mathbf{C} \times D \rightarrow \mathbf{C}$ , such that for every  $w \in P$  the equation  $f(z, w) = 0$  has no solutions, and for each  $w \in D \setminus P$  it has infinitely many solutions.*

It is not clear whether a similar result holds with multidimensional parameter space  $D$  and arbitrary closed pluripolar set  $P \subset D$ .

*Proof.* Let  $\phi : D \rightarrow D \setminus P$  be a universal covering. Let  $S$  be the set of singular points of  $\phi$  on the unit circle. Then  $S$  is a closed set of zero Lebesgue measure.

(We recall a simple proof of this fact. As a bounded analytic function,  $\phi$  has radial limits almost everywhere. It is easy to see that a point where the radial limit has absolute value 1 is not a singular point of  $\phi$ . Thus the radial limits exist and belong to  $P$  almost everywhere on  $S$ . Let  $u$  be the “Evans potential” of  $P$ , that is a harmonic function in  $D \setminus P$ , continuous in  $\overline{D}$  and such that  $u(\zeta) = 0$  for  $\zeta \in \partial D$  and  $u(\zeta) = -\infty$  for  $\zeta \in P$ . Such function exists for every compact set  $P$  of zero capacity. Now  $v = u \circ \phi$  is a negative harmonic function in the unit disc, whose radial limits on  $S$  are equal to  $-\infty$ , thus  $|S| = 0$  by the classical uniqueness theorem.)

According to a theorem of Fatou, (see, for example, [3, Ch. VI]), for every set  $S$  of zero Lebesgue measure on  $\partial D$ , there exists a holomorphic function  $g$  in  $D$ , continuous in  $\overline{D}$  and such that

$$\{\zeta \in \overline{D} : g(\zeta) = 0\} = S.$$

In particular,  $g$  has no zeros in  $D$ .

Now we define the following set  $Q \subset \mathbf{C} \times D$ :

$$Q = \{(1/g(\zeta), \phi(\zeta)) : \zeta \in D\}.$$

It is evident that the projection of  $Q$  on the second coordinate equals  $D \setminus P$ . It remains to prove that the set  $Q$  is analytic. For this, it is enough to establish that the map

$$\Phi : D \rightarrow \mathbf{C} \times D, \quad \Phi(\zeta) = (1/g(\zeta), \phi(\zeta))$$

is proper. Let  $K \subset \mathbf{C} \times D$  be a compact subset. Then the closure of  $\Phi^{-1}(K)$  in  $\overline{D}$  is disjoint from  $S$  because  $g$  is continuous and  $1/g(\zeta) \rightarrow \infty$  as  $\zeta \rightarrow S$ . On the other hand, for every point  $\zeta \in \partial D \setminus S$ , the limit

$$\lim_{\zeta \rightarrow \zeta_0} \phi(\zeta)$$

exists and has absolute value 1. So  $\Phi^{-1}(K)$  is compact in  $D$ .

Now the existence of the required function  $f$  follows from the solvability of the Second Cousin problem [4, sect. 5.6].  $\square$

Notice that the map  $\Phi$  constructed in the proof is an immersion.

We recall that a point  $a \in \mathbf{C}$  is called an exceptional value of an entire function  $f$  if the equation  $f(z) = a$  has no solutions. Picard's Little theorem says that a non-constant entire function can have at most one exceptional value.

Let  $f$  be an entire function of  $z$  depending of the parameter  $w$  holomorphically, as in Theorems 1 and 2, and

*assume in the rest of the paper that for all  $w \in D$ ,  $f(., w) \neq \text{const}$ .*

Let  $n(w) \in \{0, 1\}$  be the number of exceptional values of  $f(., w)$ .

*Question 1.* What can be said about  $n(w)$  as a function of  $w$ ?

**Example 1.**  $f(z, w) = e^z + wz$ . We have  $n(0) = 1$  and  $n(w) = 0$  for  $w \neq 0$ .

**Example 2.**  $f(z, w) = (e^{wz} - 1)/w$  for  $w \neq 0$ , and  $f(z, 0) = z$ . We have  $n(0) = 0$ , while  $n(w) = 1$  for  $w \neq 0$ . The exceptional value  $a(w) = -1/w$  tends to infinity as  $w \rightarrow 0$ .

Thus  $n$  is neither upper nor lower semicontinuous.

**Example 3.** Let

$$f(z, w) = \int_{-\infty}^z (\zeta + w)e^{-\zeta^2/2} d\zeta,$$

where the contour of integration consists of the negative ray, passed left to right, followed by a curve from 0 to  $z$ . We have  $f(z, 0) = -e^{-z^2/2}$ , which has exceptional value 0, so  $n(0) = 1$ . It is easy to see that there are no exceptional values for  $w \neq 0$ , so  $n(w) = 0$  for  $w \neq 0$ . Thus  $n(w)$  is the same as in Example 1, but this time we have an additional feature that the set of singular values of  $f(., w)$  is finite for all  $w \in \mathbf{C}$ , namely, there is one critical value  $f(-w, w)$  and two asymptotic values, 0 and  $\sqrt{2\pi}w$ .

*Question 2.* Suppose that  $n(w) \equiv 1$ , and let  $a(w)$  be the exceptional value of  $f(., w)$ . What can be said about  $a(w)$  as a function of  $w$ ?

For functions of finite order, Question 2 was addressed by Nishino [9] who proved the following:

*Let  $f$  be an entire function of two variables such that  $z \mapsto f(z, w)$  is a non-constant function of finite order for all  $w$ . If  $n(w) = 1$  for all  $w$  in some set having a finite accumulation point, then there exists a meromorphic function  $\tilde{a}(w)$  such that  $a(w) = \tilde{a}(w)$  when  $aw \neq \infty$ , and  $f(., w)$  is a polynomial when  $\tilde{a}(w) = \infty$ .*

**Example 4.** (Nishino) Let  $f(z, w) = (e^{we^z} - 1)/w$ ,  $w \neq 0$  and  $f(z, 0) = e^z$ . Then  $f$  is an entire function of two variables and  $n(w) \equiv 1$ . However  $a(w) = -1/w$ ,  $w \neq 0$  and  $a(0) = 0$ , so  $a$  is a discontinuous function of  $w$ .

Nishino also proved that for arbitrary entire function  $f$  of two variables, with  $n(w) = 1$  in some region  $D$ , the set of discontinuity of the function  $a(w)$  is closed and nowhere dense in  $D$ .

Our Theorem 3 below gives a complete answer to Question 2. We first prove the following semi-continuity property of the set of exceptional values

which holds for all *meromorphic functions* holomorphically depending on parameter. We always assume that  $z \mapsto f(z, w)$  is non-constant for all  $w$ . Denote

$$A(w) = \{a \in \overline{\mathbf{C}} : (\forall z \in \mathbf{C}) f(z, w) \neq a\}.$$

**Proposition 1.** *For every  $w_0 \in D$  and every  $\epsilon > 0$  there exists  $\delta > 0$  such that  $|w - w_0| < \delta$  implies that  $A(w)$  is contained in the  $\epsilon$ -neighborhood of  $A(w_0)$  with respect to the spherical metric.*

*Proof.* Let  $U$  be the open  $\epsilon$ -neighborhood of  $A(w_0)$ . Then  $K = \overline{\mathbf{C}} \setminus U$  is compact, so there exists  $r > 0$  such that the image of the disc  $|z| < r$  under  $f(\cdot, w_0)$  contains  $K$ . Then by Hurwitz's theorem, for every  $w$  close enough to  $w_0$  the image of the disc  $|z| < r$  under  $f(\cdot, w)$  will also contain  $K$ .  $\square$

**Corollary 1.** *The set of meromorphic functions having no exceptional values on the Riemann sphere is open in the topology of uniform convergence on compact subsets of  $\mathbf{C}$  with respect to the spherical metric in the image.*

The set of entire functions whose only exceptional value is  $\infty$  is not open as Example 2 above shows.

**Corollary 2.** *Suppose that  $f(\cdot, w)$  is entire and has an exceptional value  $a(w) \in \mathbf{C}$  for all  $w$  on some subset  $E \subset D$ . If  $a(w)$  is bounded on  $E$  then its restriction on  $E$  is continuous.*

Example 4 shows that  $a$  can be discontinuous.

**Theorem 3.** *Let  $f : \mathbf{C} \times D \rightarrow \mathbf{C}$ , be a holomorphic function, where  $D$  is a region in  $\mathbf{C}$ , and  $z \mapsto f(z, w)$  is not constant for all  $w \in D$ . Assume that for some function  $a : D \rightarrow \mathbf{C}$  we have  $f(z, w) \neq a(w)$  for all  $z \in \mathbf{C}$ .*

*Then there exists a discrete set  $E \subset D$  such that  $a$  is holomorphic in  $D \setminus E$ , and  $a(w) \rightarrow \infty$  as  $w \rightarrow w_0$ , for every  $w_0 \in E$ .*

So the singularities of  $a$  can be only of the type described in Example 4.

The main ingredient of the proof of Theorem 3 is the following recent result of N. Shcherbina [11]:

**Theorem A.** *Let  $h$  be a continuous function in a region  $G \in \mathbf{C}^n$ . If the graph of  $h$  is a pluripolar subset of  $\mathbf{C}^{n+1}$  then  $h$  is analytic.*

*Proof of Theorem 3.* Consider the analytic set

$$Q = \{(z, w, a) \in \mathbf{C}_z \times D \times \mathbf{C}_a : f(z, w) - a = 0\}.$$

Let  $R$  be the complement of the projection of  $Q$  onto  $D \times \mathbf{C}_a$ . Then it follows from the assumptions of Theorem 3 and Picard's theorem that  $R$  is the graph of the function  $w \mapsto a(w)$ . So  $R$  is a non-empty proper subset of  $D \times \mathbf{C}_a$ . Proposition 1 implies that  $R$  is closed in  $D$ .

Let us show that  $R$  is pluripolar. It is enough to show that every point  $(w_0, a_0) \in D \times \mathbf{C}_a$  has a neighborhood  $U$  such that  $R \cap U$  is pluripolar. By shifting the origin in  $\mathbf{C}_z$  and shrinking  $U$  we may assume that  $f(0, w) \neq a$  for all  $(a, w) \in U$ . Let  $r(w, a)$  be the smallest of the moduli of zeros of the entire function  $z \mapsto f(z, w) - a$ . If this function has no zeros, we set  $r(w, a) = +\infty$ . It is easy to see that  $r$  is a continuous function from  $U$  to  $(0, +\infty]$ , and  $\log r(w, a)$  is plurisuperharmonic. Thus  $R$  is pluripolar.

It follows from Proposition 1 that  $w \mapsto |a(w)|$  is lower semi-continuous, so the sets

$$E_n = \{w \in D : |a(w)| \leq n\}, \quad \text{where } n = 1, 2, 3, \dots$$

are closed. We have  $E_1 \subset E_2 \subset \dots$ , and the assumptions of Theorem 3 imply that  $D = \cup E_j$ . Let  $E_j^0$  be the interiors of  $E_j$ , then  $E_1^0 \subset E_2^0 \subset \dots$ , and the set  $G = \cup E_j^0$  is open.

We claim that  $G$  is dense in  $D$ . Indeed, otherwise there would exist a disc  $U \subset D$  which is disjoint from  $G$ . But then the closed sets  $E_j \setminus E_j^0$  with empty interiors would cover  $D$ , which is impossible by the Baire category theorem. This proves the claim.

As  $a$  is locally bounded on  $G$ , Corollary 2 of Proposition 1 implies that  $a$  is continuous in  $G$ , so by Shcherbina's theorem,  $a$  is analytic in  $G$ . It follows from the definition of  $G$  that  $a$  does not have an analytic continuation from any component of  $G$  to any boundary point of  $G$ . Our goal is to prove that  $D \setminus G$  consists of isolated points.

If  $G$  has an isolated boundary point  $w_0$  then we have

$$\lim_{w \rightarrow w_0} a(w) = \infty. \tag{3}$$

Indeed, by Proposition 1, the limit set of  $a(w)$  as  $w \rightarrow w_0$ ,  $w \in G$  consists of at most two points,  $a(w_0)$  and  $\infty$ . On the other hand, this limit set is connected. So the limit exists. If the limit is finite, then it is equal to  $a(w_0)$  and the Removable Singularity Theorem gives an analytic continuation of  $a$  to  $G \cup w_0$ , contradicting the statement above that there is no such continuation. So the limit is infinite and (3) holds.

We add to  $G$  all its isolated boundary points, thus obtaining new open set  $G'$  containing  $G$ . Our function  $a$  has a *meromorphic* continuation to  $G'$ , which we call  $\tilde{a}$ . This meromorphic continuation coincides with  $a$  in  $G$  but does not coincide at the added points  $G' \setminus G$ .

We claim that  $G'$  has no isolated boundary points. Indeed, suppose that  $w_0$  is an isolated boundary point of  $G'$ . By the same argument as above, the limit (3) exists and is infinite. Then  $\tilde{a}$  can be extended to  $w_0$  such that the extended function has a pole at  $w_0$ , but then  $w_0$  would be an isolated boundary point of  $G$  (poles cannot accumulate to a pole) so  $w_0 \in G$  by definition of  $G'$ . This contradiction proves the claim.

Let  $F'$  be the complement of  $G'$  with respect to  $D$ . Then  $F'$  is closed, nowhere dense subset of  $D$ . Furthermore,  $F'$  has no isolated points, because such points would be isolated boundary points of  $G'$ . So  $F'$  is perfect or empty. Our goal is to prove that  $F'$  is empty.

Assume the contrary, that is that  $F'$  is perfect. The closed sets  $E_n$  cover the locally compact space  $F'$ , so by the Baire category theorem one of these  $E_n$  contains a relatively open part of  $F'$ . This means that there exists a positive integer  $n$  and an open disc  $U \subset D$  intersecting  $F'$  and such that

$$|a(w)| \leq n \quad \text{for } w \in U \cap F'. \quad (4)$$

By Corollary 2 of Proposition 1, this implies that the restriction of  $a$  on  $U \cap F'$  is continuous.

We are going to prove that  $\tilde{a}$  has a continuous extension from  $G'$  to  $U \cap F'$ , and this extension agrees with the restriction of  $a$  on  $U \cap F'$ . Let  $W$  be a point of  $U \cap F'$ , and  $(w_k)$  a sequence in  $G'$  tending to  $W$ . Choosing a subsequence we may assume that there exists a limit

$$\lim_{k \rightarrow \infty} \tilde{a}(w_k), \quad (5)$$

finite or infinite. By a small perturbation of the sequence that does not change the limit of  $\tilde{a}(w_k)$  we may assume that  $w_k$  are not poles of  $\tilde{a}$  so  $a(w_k) = \tilde{a}(w_k)$ . By Proposition 1, the limit (5) can only be  $a(W)$  or  $\infty$ .

To prove continuity we have to exclude the latter case. So suppose that

$$\lim_{k \rightarrow \infty} \tilde{a}(w_k) = \infty. \quad (6)$$

Let  $C_k$  be the component of the set

$$G' \cap \{w \in U : |w - W| < 2|w_k - W|\},$$



that contains  $w_k$ , and let

$$m_k = \inf\{|\tilde{a}(w)| : w \in C_k\}. \quad (7)$$

We claim that  $m_k \rightarrow \infty$ . Indeed, suppose this is not so, then choosing a subsequence we may assume that  $m_k \leq m$  for some  $m > n$ . Then there exists a curve in  $C_k$  connecting  $w_k$  to some point  $w'_k \in C_k$  such that  $|\tilde{a}(w'_k)| \leq m+1$ . As  $|a|$  is continuous in  $C_k$ , (6) implies that this curve contains a point  $y_k$  such that  $|\tilde{a}(y_k)| = m+1$ . By selecting another subsequence we achieve that

$$\lim_{k \rightarrow \infty} \tilde{a}(y_k) = y, \quad \text{where } |y| = m+1 > n.$$

As  $|a(W)| \leq n$ , we obtain a contradiction with Proposition 1. This proves our claim that  $m_k \rightarrow \infty$  in (7).

So we can assume that

$$m_k \geq n+1 \quad \text{for all } k. \quad (8)$$

Let us show that this leads to a contradiction. Fix  $k$ , and consider the limit set of  $\tilde{a}(w)$  as  $w \rightarrow \partial C_k \cap U$  from  $C_k \cap U$ . In view of (4), (8) and Proposition 1, this limit set consists of the single point, namely  $\infty$ . To see that this is impossible, we use the following

**Lemma.** *Let  $V$  and  $C$  be two intersecting regions in  $\mathbf{C}$ , and  $g$  a meromorphic function in  $C$  such that*

$$\lim_{w \rightarrow W, w \in C} g(w) = \infty \quad \text{for all } W \in V \cap \partial C.$$

*Then  $V \cap \partial C$  consists of isolated points in  $V$  and  $g$  has meromorphic extension from  $C$  to  $V \cup C$ .*

*Proof.* By shrinking  $V$ , we may assume that  $|g(w)| \geq 1$  for  $w \in C \cap V$ . Then  $h = 1/g$  has a continuous extension from  $C$  to  $C \cup V$  by setting  $h(w) = 0$  for  $w \in V \setminus C$ . The extended function is holomorphic on the set

$$\{w \in C \cap V : h(w) \neq 0\},$$

so by Rado's theorem [10, Thm. 3.6.5],  $h$  is analytic in  $V \cup C$ ,  $1/g$  gives the required meromorphic extension of  $g$ .  $\square$

Applying this Lemma with

$$C = C_k, V = \{w \in U : |w - W| < 2|w_k - W|\}$$

and  $g = \tilde{a}$ , and taking into account that  $\partial C_k \cap V \subset F'$ , and  $F'$  has no isolated points, we arrive at a contradiction which completes the proof that  $\tilde{a}$  has a continuous extension to  $F' \cap U$ , an extension which agrees with  $a$  on  $F' \cap U$ .

From Theorem A we obtain now that  $a$  is analytic on  $F' \cap U$ , which contradicts the fact stated in the beginning of the proof that  $a$  has no analytic continuation from  $G$ .

This contradiction shows that  $F' = \emptyset$  which proves the theorem.  $\square$

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