Answers to the final exam in 530

1. $a_{m+n} = c_m a_n + \ldots + c_1 a_{m+n-1}$ for $n = 0, \ldots$ just means that, starting from the m place, the power series of f coincides with the power series of the product fg, where $g(z) = c_1 z + \ldots + c_m z^m$. So $f - f_{m-1} = fg - (fg)_{m-1}$, where $(.)_{m-1}$ denotes the $m-1$-st patrial sum. Thus

$$f = \frac{f_{m-1} - (fg)_{m-1}}{1-g}.$$

2. The integrand's power series expansion is

$$\sum_{k=0}^{\infty} \frac{(-1)^k z^{n-2k-1}}{(2k+1)!}.$$

The residue is the coefficient at z^{-1}, which corresponds to $n - 2k - 1 = -1$, so $k = n/2$. Thus the answer is: zero if n is odd, and $(-1)^{n/2} 2\pi i/(n+1)!$, if n is even.

3. It is

$$ze^{z/2} \prod_{n \neq 0} \left(1 - \frac{z}{2\pi i n}\right) \exp \frac{z}{2\pi i n}.$$

To figure out the exponential factor, denote it by g and take the logarithmic derivative

$$\frac{e^z}{e^z - 1} = \frac{1}{z} + g'(z) + \sum_{n \neq 0} \left(\frac{1}{z - 2\pi i n} + \frac{1}{2\pi i n}\right).$$

(It is NOT TRUE that $1/z$ plus the sum in the RHS is periodic!). But if we differentiate once more, the RHS becomes

$$g''(z) - \sum_{-\infty}^{\infty} \frac{1}{(z - 2\pi i n)^2},$$

and now the sum IS periodic (and tends to 0 when $z = iy \to \infty$). Thus $g'' = 0$ and $g(z) = az + b$. To see that $b = 0$, divide the product expansion by z and plug $z = 0$. To find a, rewrite the logarithmic derivative as

$$\frac{e^z}{e^z - 1} = \frac{1}{z} + a + \sum_{1}^{\infty} \frac{2z}{z^2 + 4\pi^2 n^2}.$$

The RHS has the form a plus and odd function. Representing the LHS in a similar form, we find that $a = 1/2$.

4. The region is a triangle with angles $0, \pi/2, \pi/2$. Applying $1/z$ transforms it into the half-strip $|x| < 1, y < 0$. Rotating, rescaling and shifting gives the half-strip $x < 0, 0 < y < \pi$. The exponential e^z maps this onto the upper half of the unit circle, which can be already treated as a "lune", or mapped onto the UHP by the Joukowski function. Alternatively, one can use sine or cosine, which map appropriate half-strips onto approproate half-planes.

5. The answer is $\pi/2$. Given the hint, there were few things to figure out:
a) That $2\sin^2 x = 1 - \cos 2x = 1 - \operatorname{Re} e^{2iz}$ (the integral $\int_{-\infty}^{\infty}$ of the imaginary part vanishes because this imaginary part is odd).
b) That the contribution of the great half-circle vanishes because the numerator of the integrand is bounded in the UHP, while in the denominator we have a square.
c) That the contribution of the little half-circle equals minus one half of the residue.

6. $e^{z^2/2}$.