Solutions and comments, first exam, fall 2006.

1. Find all solutions of the equation $z^3 = 8i$. The answer should be given in an algebraic form, that is using arithmetic operations and radicals of positive numbers only. No sines, exponentials, etc.

Answer: $-2i$, $\sqrt{3}+i$ and $-\sqrt{3}+i$.

Comment: correct answer in terms of the exponential was usually credited at about 5 pt.

2. Describe the set of all complex numbers z where $\cos z$ is real, and make a picture of this set.

Answer: Infinitely many vertical lines

$$\{x+iy : x = 2\pi k,\ y \in \mathbf{R}\},\ k = 0, \pm 1, \pm 2, \ldots,$$

and one horizontal line, the real axis.

3. Evaluate the integrals

$$a) \int_{\{z:|z-1|=1\}} \frac{dz}{(1-z)^3}, \quad b) \int_{\{z:|z+1|=1\}} \frac{\cos z\, dz}{(1-z)^3}, \quad c) \int_{\{z:|z-1|=1\}} \overline{z} dz.$$

Solution: a) is zero, because $(1-z)^{-3}$ has a primitive in the plane punctured at 1.

b) is zero because the function is analytic in a simply-connected region, say in the half-plane $\{x+iy : x < 1\}$, that contains the path of integration, so Cauchy's theorem is applicable.

c) equals $2\pi i$ (computation by parametrization and reducing to the ordinary integral).

Comments. Each a), b, c) was worth 3 or 4 points, depending on the correctness and clarity of explanation. Only partial credit was given for the right answer without any explanation, or with a wrong explanation.

4. Describe the set of complex numbers that satisfy the equation $|z-1| + |z+1| = 7$. Make a picture. What is $\max |z|$ for z in this set?

Solution. The equation says that the sum of distances from z to 1 and from z to -1 equals 7. This is an ellipse with foci ± 1. Its larger axis belongs to the real line, and the its extremities are $\pm 7/2$.

Comments. To get a full credit, one had to do three things: a) to describe the figure, at least to say that this is an ellipse, b) to make a plausible picture, and c) to find the maximum.

5. a) Let u and v be two harmonic functions. Is the product uv necessarily harmonic?

b) Let u and v be two harmonic functions, and v is harmonically conjugate to u. Is the product necessarily harmonic?

(Both answers should be justified: if "yes", explain why, if "no", give a counter-example).

Solution. a) No. To justify this one has to give an example. Most examples chosen at random will do, for example, take $u(x,y) = v(x,y) = x$. Then $uv = x^2$ is not harmonic.

b) Yes. One has to explain why. It is a simple computation using CR-equations:

$$(uv)_{xx} = u_{xx} + v_{xx} + 2u_x v_x$$

and

$$(uv)_{yy} = u_{yy} + v_{yy} + 2u_y v_y$$

adding these two and using that u and v are harmonic, we obtain:

$$\Delta(uv) = 2(u_x v_x + u_y v_y)$$

Replacing $u_x = v_y$ and $u_y = -v_x$ from CR-equations, we obtain that this equals to

$$v_y v_x - v_x v_y = 0.$$

Here is a much better explanation (found by some students in this exam): the function $f = u + iv$ is analytic, and $2uv$ is its imaginary part.

6. Find a harmonic function $u(z)$ in the first quadrant $\{z = x + iy : x > 0, y > 0\}$ which takes the following boundary values: -1 on the positive imaginary ray, and 1 on the positive ray.

Answer: $-\dfrac{4}{\pi}\operatorname{Arg} z + 1$.

7. (Each question is worth 2 points, no partial credit, no justification necessary).

True or false:

a) If an analytic function f in some region D has constant absolute value in D then f is constant.

True.

b) If f is an analytic in some region D, then integral of f over every closed curve in D is zero.

False. For example, integral of $1/z$ over a circle centered at the origin.

c) If f is a continuous function in some region D, and integral of f over every closed curve in D is zero, then f is analytic in D.

True. This is Morera's theorem (Theorem 18 on p. 210), proved in class.

d) If u and v are harmonic functions, and $u + iv$ is analytic then $v + iu$ is also analytic.

False. For example $x + iy$ is analytic, but $y + ix$ is not (does not satisfy the Cauchy–Riemann equations).

e) If u is a harmonic function in the whole plane and f is an analytic function in the whole plane, then $u(f(z))$ is harmonic in the whole plane.

True. Every harmonic function u in the whole plane is the real of some analytic function g. Then $u(f(z))$ is the real part of the analytic function $g(f(z))$. Composition of analytic functions is analytic where it is defined (chain rule).