First Exam Solutions

1. a) Writing z_1 and z_2 in trigonometric form $z_i = r_i e^{i\theta_i}$, we obtain

$$\mathrm{Re}(z_1\overline{z_2}) = \mathrm{Re}(r_1 r_2(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)) = r_1 r_2 \cos(\theta_1 - \theta_2),$$

the dot product of z_1 and z_2. Second solution: if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then

$$\mathrm{Re}(z_1\overline{z_2}) = x_1x_2 + y_1y_2.$$

b) Let us compute for example the angle between z_1 and z_2. We have

$$1 = |z_3|^2 = |z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\mathrm{Re}(z_1\overline{z_2}).$$

So $\mathrm{Re}(z_1\overline{z_2}) = -1/2$, so the angle between z_1 and z_2 is $2\pi/3$. Another way is to write $-z_3 = z_1 + z_2$ and to apply Cosine Theorem to show that the angle between z_1 and z_2 is $2\pi/3$.

2. a) Parametrize this line as $1 + it$, $-\infty < t < \infty$, then the image under $-z^2$ will have the parametrization

$$-1 - 2it + t^2,$$

that is $x = t^2 - 1$, $y = 2t$, or $x = y^2/4 - 1$, which describes a parabola with vertex at the point -1, and branches going right.

3. a) For example $u(x, y) = v(x, y) = x$.

b) Compute the Laplacian of this product:

$$\begin{aligned}\Delta(uv) &= u_{xx}v + 2u_xv_x + uv_{xx} + u_{yy}v + 2u_yv_y + uv_{yy} \\ &= v\Delta u + u\Delta v + 2(u_xv_x + u_yv_y) \\ &= 2(u_xv_x - v_xu_x) = 0.\end{aligned}$$

4. Let γ_1 and γ_2 be small circles about 0 and 3, oriented anticlockwise. To compute the integral over γ_1, apply the Cauchy formula for the derivative to $f(z) = (\cos z)/(z - 3)$:

$$\int_{\gamma_1} \frac{f(z)}{z^2} dz = 2\pi i f'(0) = -\frac{2\pi i}{9}.$$

Putting $g(z) = (\cos z)/z^2$ we obtain by the Cauchy formula

$$\int_{\gamma_2} \frac{g(z)}{z - 3} dz = 2\pi i g(3) = \frac{2\pi i \cos 3}{9}.$$

If a curve γ makes m turns around 0 and n turns around 3 then the integral is equal to $2\pi i(n\cos 3 - m)/9$.
5. We have $f(z) = \sqrt{|z|}\exp(i(\mathrm{Arg})z/2)$. A parametrization of the upper half of the unit circle from 1 to -1 is $z = e^{it}$, $0 \le t \le \pi$, so $dz = ie^{it}$, and our integral equals

$$\int_0^\pi \frac{ie^{it}}{e^{it/2}}dt = i\int_0^\pi e^{it/2}dt = 2(i-1).$$

6. The equation $\sin z = 2$ is equivalent to $e^{iz} - e^{-iz} = 4i$. Denote $e^{iz} = w$, then

$$w - w^{-1} = 4i, \quad w^2 - 4iw - 1 = 0, \quad w_{1,2} = 2i \pm i\sqrt{3}.$$

For the first series we have

$$z_{1,k} = (1/i)\log((2+\sqrt{3})i) = \pi/2 + 2k\pi - i\mathrm{Log}(2+\sqrt{3}), \quad k = 0, \pm 1, \pm 2, \ldots.$$

Similarly,

$$z_{2,k} = (1/i)\log((2-\sqrt{3})i) = \pi/2 + 2k\pi - i\mathrm{Log}(2-\sqrt{3}), \quad k = 0, \pm 1, \pm 2, \ldots.$$

To make an accurate sketch, notice that $\mathrm{Log}(2+\sqrt{3})$ and $\mathrm{Log}(2-\sqrt{3})$ have opposite signs and equal absolute values, because $(2-\sqrt{3})(2+\sqrt{3}) = 1$.