Some answers and solutions to the practice problems

1. a) 1; b) 1; c) 6.

2. a) Let

$$f(z) = \frac{e^{iz}}{z^2 - 2iz - 2}.$$

the poles are

$$z_{1,2} = i \pm 1,$$

both in the upper half-plane. The formula for such integrals that we learned gives

$$I = 2\pi i \left(\mathrm{res}_{z_1} f(z) + \mathrm{res}_{z_2} f(z)\right).$$

Computing the residues, we obtain

$$I = -\frac{2\pi \sin 1}{e}.$$

b) This integral is actually equal to 0 since

$$\int_0^1 \frac{\operatorname{Log} x}{(x+1)^2} dx = -\int_1^\infty \frac{\operatorname{Log} x}{(x+1)^2} dx,$$

which can be seen by the change of the variable $x \mapsto 1/x$.

But the regular approach would be to integrate

$$f(z) = \frac{\operatorname{Log}_0^2 z}{(z+1)^2}$$

over the boundary of slit disks $\{z : |z| < R, \, 0 < \arg z < 2\pi\}$, where R is large, compute the residue at -1, and use the value of

$$\int_0^\infty \frac{dx}{(x+1)^2} = 1,$$

which is easy to obtain by calculus.

c) Let $f(z) = (1 - e^{2iz})/z^2$. This has a *simple* pole at 0, while the function in the original integral has removable singularity. Then we have

$$\int_0^\infty \frac{1 - \cos(2x)}{x^2} dx = \frac{1}{2} \mathrm{Re}\, p.v. \int_{-\infty}^\infty f(z) dz$$

$$= \frac{1}{2}\pi i \,\mathrm{res}_0 f(z) = \pi.$$

3. $-\pi^2/12$ (I solved this in class).

4. a) Let $w = z - 3$. Then

$$\frac{1}{z-2} = \frac{1}{1+w} = w^{-1}\frac{1}{1+w^{-1}} = \sum_0^\infty (-1)^n (z-3)^{-n-1}.$$

b)

$$z^2 \sin \pi \frac{z+1}{z} = z^2 \sin(\pi + \pi/z) = -z^2 \sin(\pi/z)$$
$$= \sum_0^\infty (-1)^{n+1} \frac{\pi^{2n+1}}{(2n+1)!} z^{1-2n}.$$

5. a) every integer is a removale singularity, ∞ is not isolated.
   b) $\pi/2 + \pi k$ is essential, for all integer k; ∞ is not isolated.
   c) $\pi i + 2\pi i k$ is a simple pole for every integer k, ∞ not isolated.

6. a) 3; b) $\sqrt{2}$; c) 1.

7. a)

$$\frac{1}{z^2-1} = z^{-2}\frac{1}{1-z^{-2}} = \sum_0^\infty z^{-2n-2}.$$

b)

$$\frac{1}{z^2-2} = \frac{1}{((z+1)-1)^2-1} = \frac{1}{(z+1)^2-2(z+1)}$$
$$= -\frac{1}{2(z+1)}\frac{1}{1-(z+1)} = -\frac{1}{2}\sum_0^\infty (z+1)^{n-1}.$$

c)

$$\sum_0^\infty \frac{(-1)^n}{(2n+1)!} z^{-2n+1}.$$

8. $e^x(x \sin y + y \cos y)$.

9. a) yes; b) no; c) yes; d) no; e) no; f) yes.