Fourier transform in $\mathbf{R}^n$

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1. Definitions and simple properties.

We use boldface letters to denote vectors $\mathbf{x} = (x_1, \ldots, x_n)$ in $\mathbf{R}^n$, and

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \ldots + x_n y_n$$

is the usual dot product, and $|\mathbf{x}| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$ is the norm of the vector.

$$\int f(\mathbf{x}) d\mathbf{x} \quad \text{means} \quad \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_n) dx_1 \ldots dx_n.$$

In polar coordinates, $(n = 2)$

$$\int f(\mathbf{x}) d\mathbf{x} = \int_{-\pi}^{\pi} \int_{0}^{\infty} f(r \cos \theta, r \sin \theta) r dr d\theta.$$

In spherical coordinates $(n = 3)$

$$\int f(\mathbf{x}) d\mathbf{x} = \int_{-\pi}^{\pi} \int_{0}^{\pi} \int_{0}^{\infty} f(r \sin \phi \cos \theta, r \sin \phi \sin \theta, r \cos \phi) r^2 \sin \phi \, dr \, d\phi \, d\theta.$$

The integrals over the whole space should be convergent. For example,

$$\int_{|\mathbf{x}| \geq 1} |\mathbf{x}|^{-\alpha} d\mathbf{x} < \infty$$

when $\alpha > n$ and divergent when $\alpha < n$. And

$$\int_{|\mathbf{x}| \leq 1} |\mathbf{x}|^{-\alpha} d\mathbf{x} < \infty$$

when $\alpha < n$ and divergent when $\alpha > n$.

We will use multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, where α_j are positive integers, and

$$|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n.$$

For example,

$$\mathbf{x}^\alpha := x_1^{\alpha_1} \ldots x_n^{\alpha_n}$$

and

$$\frac{\partial^{|\alpha|} f}{\partial \mathbf{x}^\alpha} := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}.$$

Spaces $L^2(\mathbf{R}^n)$, $L^1(\mathbf{R}^n)$ and the Schwartz space $\mathscr{S}$ are defined as usual, for example, $f \in \mathscr{S}$ means

$$\sup_{\mathbf{R}^n} \left| \frac{f^{|\alpha|}}{\partial \mathbf{x}_\alpha}(\mathbf{x}) \right| (1 + |\mathbf{x}|^m) < \infty,$$

for all α and m.

Fourier transform is defined for $f \in \mathscr{S}$ by the formula

$$F[f](s) := \hat{f}(\mathbf{s}) = \int f(\mathbf{x}) e^{-i\mathbf{s}\cdot\mathbf{x}} d\mathbf{x}.$$

Here $\mathbf{s} = (s_1, \ldots, s_n)$ is a vector, and it is the dot product that stands in the exponential).

Its properties are almost completely analogous to the properties we established earlier for $n = 1$:

1. Fourier transform is a linear operation.
2. $F[f(\mathbf{x} - \mathbf{a})] = e^{-i\mathbf{a}\cdot\mathbf{s}} \hat{f}(\mathbf{s}), \quad F[e^{i\mathbf{a}\cdot\mathbf{x}} f(\mathbf{x})] = \hat{f}(\mathbf{s} - \mathbf{a}).$
3. $F[f(\delta \mathbf{x})] = \delta^{-n} \hat{f}(\delta^{-1}\mathbf{s}), \quad F[\delta^{-n} f(\delta^{-1}\mathbf{x})] = \hat{f}(\delta \mathbf{s}), \quad \delta > 0.$
4. Differentiation and multiplication rules

$$F\left[\left(\frac{\partial}{\partial \mathbf{x}} \right)^\alpha f(\mathbf{x}) \right] = (i\mathbf{s})^\alpha \hat{f}(\mathbf{s}),$$

$$F[\mathbf{x}^\alpha f(\mathbf{x})] = i^{|\alpha|} \left(\frac{\partial}{\partial \mathbf{s}} \right)^\alpha f(\mathbf{s})$$

5. Convolution rules

$$F(f \star g) = \hat{f}\hat{g}, \quad F[fg] = (2\pi)^{-n}\hat{f} \star \hat{g}.$$

6. Inversion formula

$$f(\mathbf{x}) = \frac{1}{(2\pi)^n} \int e^{i\mathbf{s}\cdot\mathbf{x}} \hat{f}(\mathbf{s}) d\mathbf{s}.$$

7. Plancherel's formula

$$(\hat{f}, \hat{g}) = (2\pi)^n (f, g), \quad \|\hat{f}\|^2 = (2\pi)^n \|f\|^2.$$

To state an additional rule, I recall that an $n \times n$ matrix represents a rotation of the space if it is orthogonal and has determinant 1, that is

$$A^T A = I, \quad \text{and} \quad \det(A) = 1,$$

and rotations preserve the dot product

$$(A\mathbf{x}, A\mathbf{y}) = (x, y).$$

Then we easily obtain:

8. Fourier transform commutes with rotations: $F[f(A\mathbf{x})] = \hat{f}(A\mathbf{s})$.

Using these rules, one easily generalizes solutions of the heat equation in the whole space, and Laplace equation in the half-space. The heat kernel is

$$K_n(\mathbf{x}, t) = (4\pi kt)^{-n/2} \exp\left(-\frac{|\mathbf{x}|^2}{4kt}\right),$$

and Poisson's kernel for the upper half-space

$$\{(\mathbf{x}, y) : \mathbf{x} \in \mathbf{R}^n, y > 0\} \subset \mathbf{R}^{n+1}$$

is

$$\frac{\Gamma((n+1)/2))}{\pi^{(n+1)/2}} \frac{y}{(|\mathbf{x}|^2 + y^2)^{(n+1)/2}},$$

which is the inverse transform of $\exp(-|\mathbf{s}|y)$.

However there are interesting differences when we consider the wave equation in $\mathbf{R}^n$: the behavior of solution is very different in different dimensions.

For the Fourier transform of the solution we obtain the formula which looks almost exactly the same as in dimension 1:

$$\hat{u}(\mathbf{s},t) = \hat{f}(\mathbf{s})\cos(ct|\mathbf{s}|) + \hat{g}(\mathbf{s})\frac{\sin(ct|\mathbf{s}|)}{c|\mathbf{s}|}. \tag{1}$$

but there are difficulties with finding the inverse transform.

Before we address this question, let us consider another one:

Radial functions. These are functions with the property $f(R\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x}$ and all rotations R. It is easy to understand that such functions depend only on $r = |\mathbf{x}|$, so we will write $f(\mathbf{x}) = f_0(|\mathbf{x}|)$. It follows from property 8 of Fourier transform that Fourier transform of a radial function is also radial. Sos is $f(\mathbf{x}) = f_0(|\mathbf{x}|)$ we have $\hat{f}(\mathbf{s}) = g_0(|\mathbf{s}|)$, and we want to obtain a formula for g_0. The formula will depend on the dimension n.

$n = 2$. We have in polar coordinates $\mathbf{x} = (r\cos\theta, r\sin\theta)$, $\mathbf{s} = (\rho\cos\phi, \rho\sin\phi)$,

$$\begin{aligned}\hat{f}(\mathbf{s}) &= \int f(\mathbf{x})e^{-i\mathbf{s}\cdot\mathbf{x}}d\mathbf{x}\\ &= \int_0^{2\pi}\int_0^{\infty} f_0(r)e^{-ir\rho\cos(\phi-\theta)}r\,dr\,d\theta.\end{aligned}$$

Let us change the variable θ to t, where $\theta = \phi + t - \pi/2$. Then the integral with respect to t will be

$$\int_0^{2\pi} e^{ir\rho\sin t}dt = 2\pi J_0(r\rho),$$

where J_0 is the Bessel function, see “Bessel functions”, section 3, formula (24). Thus

$$\hat{f}(\mathbf{s}) = g_0(\rho) = 2\pi\int_0^{\infty} f_0(r)J_0(r\rho)r\,dr, \quad \rho = |\mathbf{s}|.$$

This is called the *Hankel transform* of order 0.

Exercise. Prove the inversion formula for the Hankel transform

$$f_0(r) = \frac{1}{2\pi}\int_0^{\infty} g_0(\rho)J_0(r\rho)\rho\,d\rho.$$

$n = 3$. It turns out that this case is simpler. In spherical coordinates, we write

$$\mathbf{x} = (r \sin\phi \cos\theta, r \sin\phi \sin\theta, r \cos\phi)$$

and $|\mathbf{s}| = \rho$. Since we already know that $\hat{f}$ depends only on ρ, it is sufficient to compute it at points $\mathbf{s} = (0, 0, \rho)$, so $\mathbf{s} \cdot \mathbf{x} = r\rho \cos\phi$ and

$$g_0(\rho) = \hat{f}(0, 0, \rho) = \int_0^{2\pi} \int_0^{\infty} \int_0^{\pi} f_0(r) e^{-ir\rho\cos\phi} r^2 \sin\phi \, d\phi \, dr \, d\theta. \tag{2}$$

Unlike in dimension 2, we have an elementary integral with respect to ϕ:

$$\int_0^{\pi} e^{-ir\rho\cos\phi} \sin\phi \, d\phi = \int_{-1}^{1} e^{ir\rho u} du = \frac{1}{ir\rho} e^{ir\rho u}|_{-1}^{1} = 2\frac{\sin(r\rho)}{r\rho}. \tag{3}$$

Thus

$$g_0(\rho) = \frac{4\pi}{\rho} \int_0^{\infty} f_0(r) \sin(r\rho) r \, dr. \tag{4}$$

Exercise. Prove the inversion formula

$$f_0(r) = \frac{1}{2\pi^2 r} \int_0^{\infty} g_0(\rho) \sin(r\rho) \rho d\rho.$$

These calculations can be generalized to arbitrary dimension, and the general rule is that in odd dimensions radial Fourier transform is simpler than in even dimensions. This is related to the fact that Bessel functions of half-integer order are elementary. Radial Fourier transform in dimension n is expressed in terms of Bessel functions of order $n/2 - 1$.

Now we return to the wave equation, and to inversion of the Fourier transform (1).

Case $n = 3$. First we notice that

$$\cos(ct|\mathbf{s}|) = \frac{d}{dt} \frac{\sin(ct|\mathbf{s}|)}{c|\mathbf{s}|}, \tag{5}$$

so it is enough to invert the second summand in the formula (1). Here, unlike in dimension 1, we are confronted with the difficulty that the function $(\sin|\mathbf{s}|)/|\mathbf{s}|$ is not integrable.

Let us consider the average of the function $e^{-i\mathbf{s}\cdot\gamma}$ on the unit sphere. We denote the area element on the unit sphere by $d\sigma(\gamma)$. Here

$$\gamma = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

is a point on the unit sphere, and $d\sigma(\gamma) = \sin\phi\, d\phi\, d\theta$, the area element. Then we have

$$\frac{1}{4\pi}\int_{S^2} e^{-i\mathbf{s}\cdot\gamma} d\sigma(\gamma) = \frac{\sin|\mathbf{s}|}{|\mathbf{s}|}. \tag{6}$$

Here $S^2 = \{\mathbf{x} : |\mathbf{x}| = 1\}$ is the unit sphere. The meaning of the LHS is the average over the unit sphere. This formula was actually just proved by computation (3) where $r = 1$.

Now we introduce the *averaging operator* $f \mapsto M_t[f]$,

$$M_t[f](\mathbf{x}) = \frac{1}{4\pi}\int_{S^2} f(\mathbf{x} - t\gamma) d\sigma(\gamma).$$

In words: the value of the function $M_t[f]$ at the point $\mathbf{x}$ is the average of the values of f on the sphere of radius t centered at $\mathbf{x}$. (The area element on this sphere is $t^2 d\sigma$ while the area of the whole sphere is $4\pi t^2$.)

Let us express the Fourier transform of $M_t[f]$ in terms of $\hat{f}$. We have

$$F[M_t[f]](\mathbf{s}) = \frac{1}{4\pi}\int_{\mathbf{R}^3}\left(\int_{S^2} f(\mathbf{x} - t\gamma) d\sigma(\gamma)\right) e^{-i\mathbf{s}\cdot\mathbf{x}} d\mathbf{x}.$$

We change the variable to $\mathbf{y} = \mathbf{x} - \gamma t$, and use the integral (6) and obtain

$$F[M_t[f]](\mathbf{s}) = \hat{f}(\mathbf{s})\frac{\sin(|\mathbf{s}|t)}{|\mathbf{s}|t}.$$

This is the second term in (1) with $c = 1$, divided by t. Taking into account (5), we obtain the solution of the wave equation $u_{tt} = \Delta u$ in the form

$$u(x,t) = \frac{\partial}{\partial t}\left(tM_t[f](\mathbf{x})\right) + tM_t[g](\mathbf{x}). \tag{7}$$

It remains to notice that if u satisfies $u_{tt} = \Delta u$, then $v(x,t) = u(x, ct)$ satisfies $v_{tt} = c^2\Delta v$, with $v(x,0) = u(x,0),\ v_t(x,0) = cu(x,0)$, so the formula for the general case is obtained from (7) by changing t to ct, and dividing g on c. So the final answer is

$$u(\mathbf{x},t) = \frac{\partial}{\partial t}\left(tM_{ct}[f](\mathbf{x})\right) + tM_{ct}[g](\mathbf{x}). \tag{8}$$

Case $n = 2$**.** This can be reduced to the previous case $n = 3$ by the simple argument of "descent". Indeed, suppose that u solves $u_{tt} = \Delta u$ in $\mathbf{R}^2$ with initial conditions

$$u(\mathbf{x}, 0) = f(\mathbf{x}), \quad u_t(\mathbf{x}, 0) = g(\mathbf{x}), \quad \mathbf{x} \in \mathbf{R}^2.$$

Then we can extend u, f, g to functions in $\mathbf{R}^3$ simply by defining $\tilde{u}(x_1, x_2, x_3) = u(x_1, x_2)$ and similarly for f, g. Then $\tilde{u}$ will solve 3-dimensional wave equation of the same form and satisfy the initial conditions $\tilde{u}(x, 0) = \tilde{f}(\mathbf{x}),\ \tilde{u}_t(\mathbf{x}, 0) = \tilde{g}(\mathbf{x})$. Thus we only need to see what happens to the averages M_t in formula (7) when it is applied to such extensions.

It turns out that

$$M_t[\tilde{f}](x_1, x_2, 0) = \widetilde{M_t}[f](x_1, x_2), \tag{9}$$

where

$$\widetilde{M_t}[f](\mathbf{x}) = \frac{1}{2\pi} \int_{|\mathbf{y}| \leq 1} f(\mathbf{x} + t\mathbf{y})(1 - |\mathbf{y}|^2)^{-1/2} d\mathbf{y}, \quad \mathbf{x} = (x_1, x_2, 0) \in \mathbf{R}^2. \tag{10}$$

Or, in polar coordinates $\mathbf{y} = (\rho \cos\theta, \rho \sin\theta)$,

$$\widetilde{M_t}[f](\mathbf{x}) = \frac{1}{2\pi} \int_0^{\pi} \int_0^1 f\left(\mathbf{x} + (t\rho\cos\theta, t\rho\sin\theta)\right)(1 - \rho^2)^{-1/2} \rho\, d\rho\, d\theta, \quad \mathbf{x} \in \mathbf{R}^2. \tag{11}$$

To obtain this result, we compute: $\rho = |\mathbf{y}| = \sin\phi$, where $\mathbf{y}$ is the projection of the point γ on the unit sphere onto the unit disk in the plane. Then $d\rho = \cos\phi\, d\phi$, and

$$d\phi = \frac{d\rho}{\cos\phi} = \frac{d\rho}{\sqrt{1 - \rho^2}},$$

so

$$d\sigma = \sin\phi\, d\phi\, d\theta = \frac{\rho\, d\rho\, d\theta}{\sqrt{1 - \rho^2}} = \frac{dy}{\sqrt{1 - |\mathbf{y}|^2}},$$

and we obtain the result (10), (11). The factor $1/(4\pi)$ changes to $1/(2\pi)$ because the projection of the sphere on the disk is 2-to-1. So the final formula for solution of $u_{tt} = c^2 u_{xx}$ in $\mathbf{R}^2$ is

$$u(\mathbf{x}, t) = \frac{\partial}{\partial t}\left(t\widetilde{M_{ct}}[f](\mathbf{x})\right) + t\widetilde{M_{ct}}[g](\mathbf{x}). \tag{12}$$

Discussion. Let us compare the d'Alembert formula

$$u(x,t) = (f(x-ct) + f(x+ct))/2 + \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)du$$

with formulas (8) and (12). Suppose that something happens (a flash) near $(\mathbf{x}, t) = (0, 0)$. This means that the initial conditions f, g are supported by a small neighborhood of 0. If the observer is placed at a point $\mathbf{x}$, she will see nothing until the time $t = |\mathbf{x}|/c$, that is $u(\mathbf{x}, t) = 0$ for $t < |\mathbf{x}|/c$. This means that the wave has a finite speed of propagation c, and this is a property of the wave equation in all dimensions. Notice that heat equation does not have this property: with any positive initial condition solution will be positive in the whole space at all times.

Now, formula (8) involves only averages over the sphere of radius ct about x. This means that when $t > |\mathbf{x}|/c$ the observer also sees nothing. So a short flash arrives to the observer at time $|\mathbf{x}|/c$ as a short flash. Solution (12) in dimension 2 is quite different: it involves integrals over the whole ball centered at $\mathbf{x}$ of radius ct. This means that for all times $t > |\mathbf{x}|/c$ the observer still sees something. If the equation described sound waves, this can be interpreted as a long echo.

So if the dimension of our space were even, we would be unable to communicate using light or sound: the whole space would be filled by continuous echo from all sources.