SPECTRAL LOCI OF STURM–LIOUVILLE OPERATORS WITH POLYNOMIAL POTENTIALS

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1. We consider differential operators

$$L : y \mapsto -y'' + Py,$$

where the potential P is a polynomial of degree d. When $d \in \{0, 1, 2\}$, the general solution of $Ly = 0$ can be expressed in terms of special functions (elementary, Airy or Weber functions, respectively). The eigenvalue problem for $d = 2$ (harmonic oscillator) plays an important role in quantum mechanics.

We mostly consider the cases $d = 3$ and $d = 4$.

Cubic and quartic oscillators were studied a lot from the very beginning of quantum mechanics, mostly by perturbative methods. Cubic oscillator arises in quantum field theory (Zinn-Justin) and in the theory of Painleve equations (Masoero).

2. Boundary conditions. By an affine transformations of the independent variable we normalize: $P(z) = z^d + O(z^{d-2})$.

Consider the sectors

$$S_j = \left\{ z : \left| \arg z - \frac{2\pi j}{d+2} \right| < \frac{\pi}{d+2} \right\},\ 0 \leq j \leq d+1.$$

Every solution y of $Ly = 0$ is an entire function. For each $j \in \{0, \ldots, d+1\}$ it either grows exponentially along all rays from the origin in S_j or tends to zero exponentially along every such ray in S_j. We choose two non-adjacent sectors, and impose the condition that the eigenfunction tends to 0 in these sectors.

With such boundary conditions, the problem has infinite discrete spectrum and eigenvalues tend to infinity. To each eigenvalue corresponds one-dimensional eigenspace.

3. Suppose that the polynomial potential depends analytically on a parameter $a \in \mathbf{C}^n$. Then the spectral locus Z is defined as the set of all pairs $(a, \lambda) \in \mathbf{C}^{n+1}$ such that the differential equation

$$-y'' + P(z, a)y = \lambda y$$

has a solution y satisfying the boundary conditions. Spectral locus is an analytic hypersurface in $\mathbf{C}^{n+1}$; it is the zero-set of an entire function $F(a, \lambda)$ which is called the spectral determinant.

The multi-valued function $\lambda(a)$ defined by $F(a, \lambda) = 0$ has the following property: its only singularities are algebraic ramification points, and there are finitely many of them over every compact set in the a-space (EG1).

Next we discuss connectedness of the spectral locus.

Theorem 1 *For the cubic oscillator*

$$-y'' + (z^3 - az + \lambda)y = 0,\ y(\pm i\infty) = 0,$$

the spectral locus is a smooth irreducible curve in $\mathbf{C}^2$*.*

Theorem 2 (EG1) *For the even quartic oscillator*

$$-y'' + (z^4 + az^2)y = \lambda y,\ y(\pm\infty) = 0,$$

the spectral locus consists of two disjoint smooth irreducible curves in $\mathbf{C}^2$*, one corresponding to even eigenfunctions, another to odd ones.*

These theorems can be generalized to polynomials of arbitrary degree if we use all coefficients as parameters (Habsch, Alexandersson).

However if we consider a subfamily of the family of all polynomials of given degree, then the spectral locus can be reducible in an interesting way.

4. Quasi-exactly solvable quartic L_J

$$-y'' + (z^4 - 2bz^2 + 2Jz)y = \lambda y,\ y(re^{\pm\pi i/3}) \to 0.$$

When J is a positive integer, this problem has J elementary eigenfunctions of the form $p(z)\exp(z^3/3 - bz)$, with a polynomial p. The (b, λ) corresponding to these eigenfunctions form the quasi-exactly solvable part Z_J^{QES} of the spectral locus Z_J, which is an algebraic curve.

Theorem 3 *Z_J^{QES} is a smooth irreducible curve in $\mathbf{C}^2$.*

Similar phenomenon occurs in degree 6: there are one parametric families of quasi-exactly solvable sextics, and for each such family the quasi-exactly solvable part of the spectral locus is a smooth irreducible algebraic curve.

When $J \to \infty$, an appropriate rescaling of Z_J^{QES} tends to the spectral locus of one-parametric cubic family, and a rescaling of the sextic QES spectral locus tends to the spectral locus of the even quartic family.

5. Hermitian and PT-symmetric operators. An eigenvalue problem can be preserved by a symmetry with respect to a line in the complex z-plane. Without loss of generality, we can take this line to be real line, and the symmetry to be the complex conjugation. Two cases are possible:

a) Each of the two boundary conditions is preserved by the symmetry. In this case the problem is Hermitian.

b) The two boundary conditions are interchanged by the symmetry. Such problems are called PT-symmetric. (Physicists prefer to choose the symmetry with respect to the imaginary line in this case. PT stands for "parity and time".)

Thus we consider a real potential P, and the boundary conditions are imposed on the real line in the Hermitian case, or are interchanged by the complex conjugation in the PT-symmetric case. For example, there is a real-one-parametric family of PT-symmetric cubics, when parameter a is real and normalization is on the imaginary line as above. There are two different real-two-parametric families of PT-symmetric quartics which we call I and II:

$$-y'' + (-z^4 + az^2 + cz + \lambda)y = 0,\ y(\pm i\infty) = 0, \tag{1}$$

and

$$-y'' + (z^4 - 2bz^2 + 2Jz)y = \lambda y, y(re^{\pm\pi i/3}) \to 0. \tag{2}$$

We begin with the cubic PT-symmetric spectral locus

$$-w'' + (z^3 - az + \lambda) = 0, \quad w(\pm i\infty) = 0. \quad (3)$$

Theorem 4 *For every integer $n \geq 0$, there exists a simple curve $\Gamma_n \subset \mathbf{R}^2$, which is the image of a proper analytic embedding of a line, and which has these properties:*
(i) For every $(a, \lambda) \in \Gamma_n$ problem (3) has an eigenfunction with $2n$ non-real zeros.
(ii) The curves Γ_n are disjoint and the real spectral locus of (3) is $\bigcup_{n \geq 0} \Gamma_n$
(iii) The map

$$\Gamma_n \cap \{(a, \lambda) : a \geq 0\} \to \mathbf{R}_{\geq 0},$$

$(a, \lambda) \mapsto a$ is a 2-to-1 covering.
(iv) For $a \geq 0$, $(a, \lambda) \in \Gamma_n$ and $(a, \lambda') \in \Gamma_{n+1}$ imply $\lambda' > \lambda$.

The following computer-generated plot of the real spectral locus of (3) is taken from Trinh's thesis (2002). Theorem 4 rigorously establishes some features of this picture.

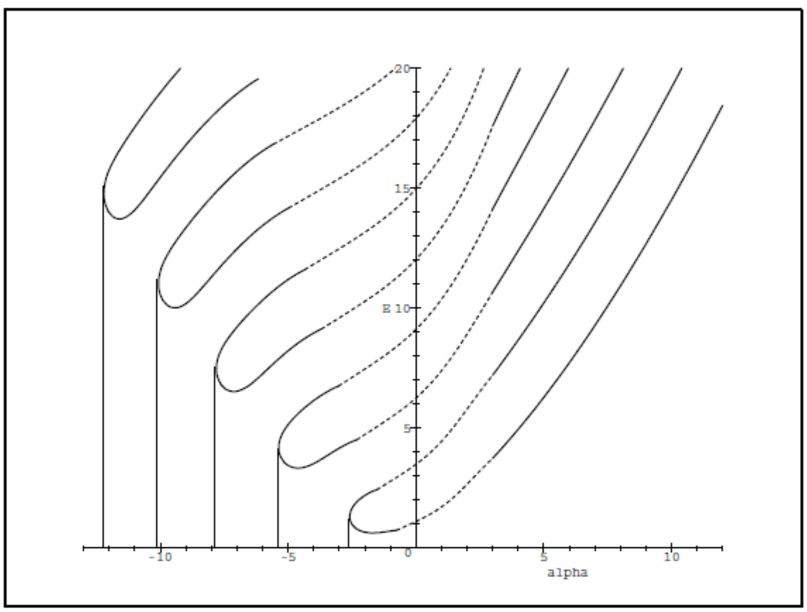

Fig 1. Real spectral locus for PT-symmetric cubic.

Consider the PT-symmetric quartic family of type I:

$$-w'' + (-z^4 + az^2 + cz)w = -\lambda w,\ w(\pm i\infty) = 0. \tag{4}$$

It is equivalent to the PT-symmetric family

$$-w'' + (z^4 + az^2 + icz)w = \lambda w, \quad w(\pm\infty) = 0,$$

studied by Bender, *et al* (2001) and Delabaere and Pham (1998).

Theorem 5 *The real spectral locus of (4) consists of disjoint smooth analytic properly embedded surfaces $S_n \subset \mathbf{R}^3$, $n \geq 0$, homeomorphic to a punctured disk. For $(a, c, \lambda) \in S_n$, the eigenfunction has exactly $2n$ non-real zeros. For large a, projection of S_n on the (a, c) plane approximates the region $9c^2 - 4a^3 \leq 0$.*

Numerical computation suggests that the surfaces have the shape of infinite funnels with the sharp end stretching towards $a = -\infty$, $c = 0$, and that the section of S_n by every plane $a = a_0$ is a closed curve.
Theorem 5 implies that this section is compact for large a_0.
The following computer-generated plot is taken from Trinh's thesis:

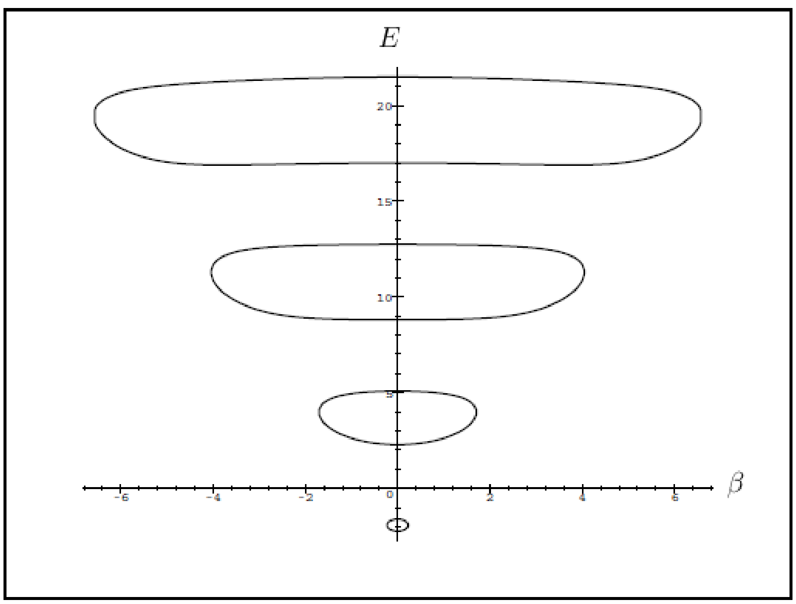

Fig 2. Section of the surfaces $S_0, \dots, S_3$ by the plane $a = -9$.

The PT-symmetric quartic family of the second type is more complicated, due to the presence of the QES spectrum. Let $Z_J^{QES}(\mathbf{R})$ be the real QES spectral locus of the operator L_J,

$$-y'' + (z^4 - 2by^2 + 2Jz)y = \lambda y, \; y(re^{\pm \pi i/3}) \to 0.$$

Theorem 6 *For $J = n + 1 > 0$, $Z_{n+1}^{QES}(\mathbf{R})$ consists of $[n/2] + 1$ disjoint analytic curves $\Gamma_{n,m}, \; 0 \leq m \leq [n/2]$.*

For $(b, \lambda) \in \Gamma_{n,m}$, the eigenfunction has n zeros, $n - 2m$ of them real.

If n is odd, then $b \to +\infty$ on both ends of $\Gamma_{m,n}$. If n is even, the same holds for $m < n/2$, but on the ends of $\Gamma_{n,n/2}$ we have $b \to \pm\infty$.

If $(b, \lambda) \in \Gamma_{n,m}$ and $(b, \mu) \in \Gamma_{n,m+1}$ and b is sufficiently large, then $\mu > \nu$.

It follows from these theorems that in each family, there are infinitely many parameter values where pairs of real eigenvalues collide and escape from the real line to the complex plane.

In the quartic family of the second type, another interesting feature of the real spectral locus is present: for some parameter values the QES spectral locus crosses the rest of the spectral locus. This is called "level crossing".

Theorem 7 *The points* $(b, \lambda) \in Z_J^{QES}$ *where the level crossing occurs are the intersection points of* Z_J^{QES} *with* Z_{-J}*. For each* $J \geq 1$ *there are infinitely many such points, in general, complex. When* J *is odd, there are infinitely many level crossing points with* $b_k < 0$ *and real* λ_k*. We have*

$$b_k \sim -((3/4)\pi k)^{2/3},\ k \to \infty.$$

The only known general result of reality of eigenvalues of PT-symmetric operators is a theorem of K. Shin, which for our quartic of second type implies that all eigenvalues are real if $J \leq 0$.

We have the following extensions of this result.

Theorem 8 *For every positive integer J, all non-QES eigenvalues of L_J are real.*

and

Theorem 9 *All eigenvalues of L_J are real for every real $J \leq 1$ (not necessarily integer).*

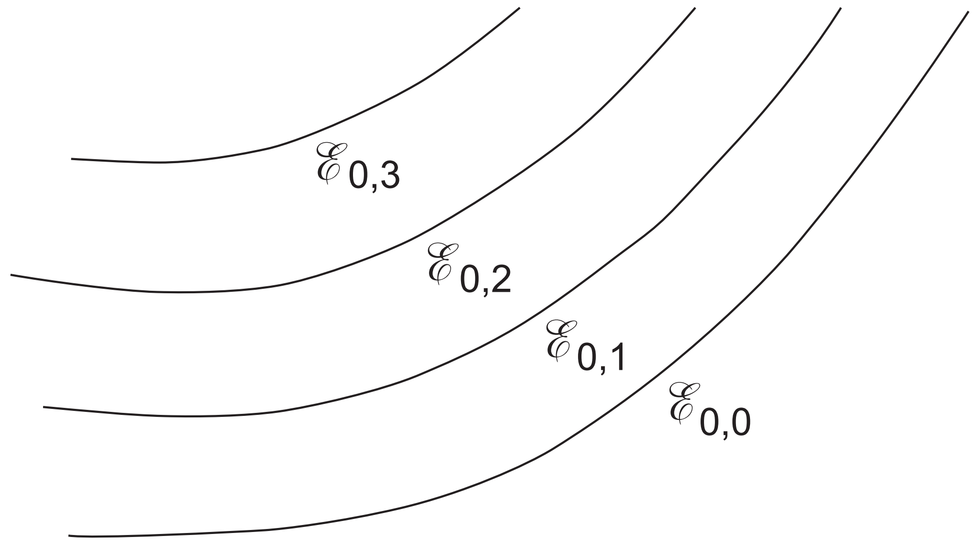

Fig. 3. $Z_0(\mathbf{R})$.

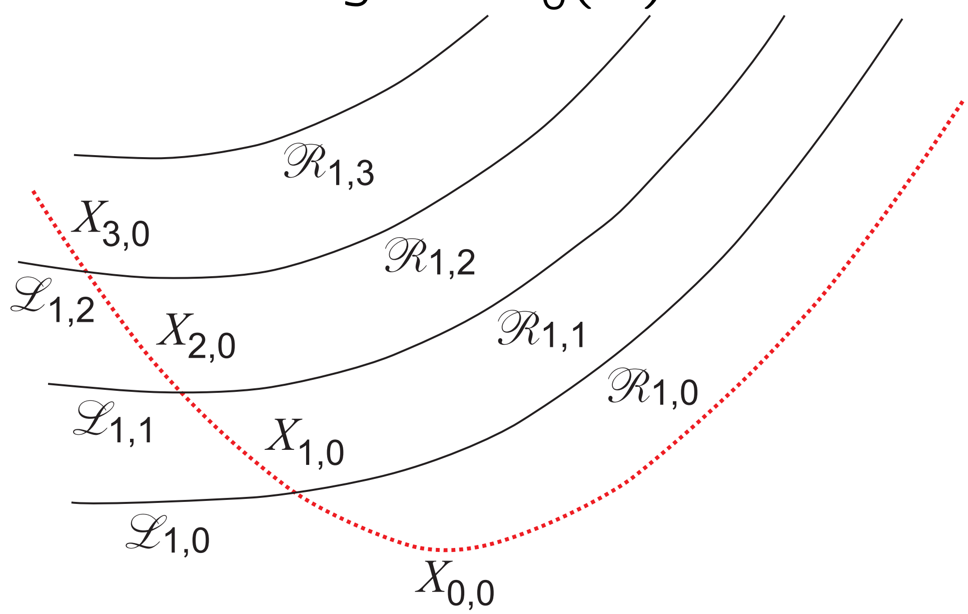

Fig. 4. $Z_1(\mathbf{R})$.

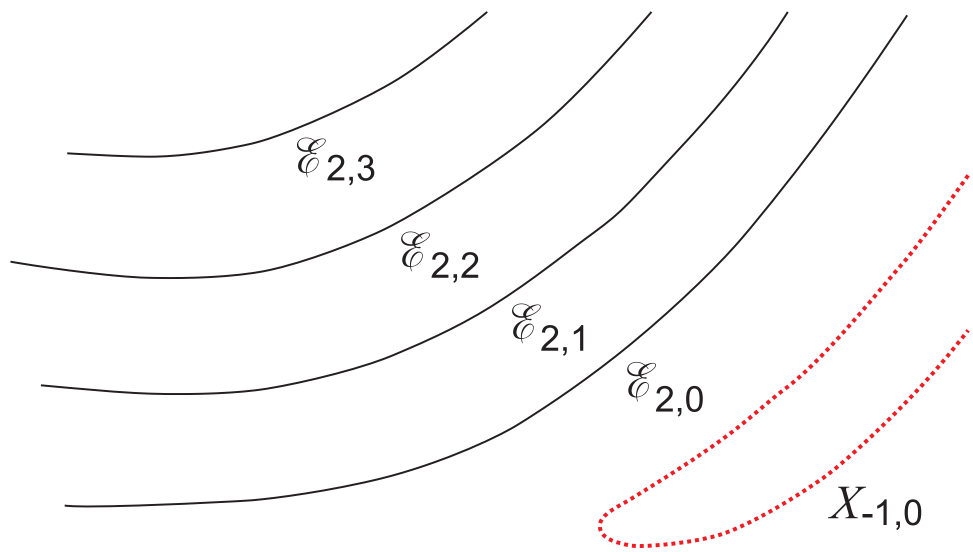

Fig. 5. $Z_2(\mathbf{R})$.

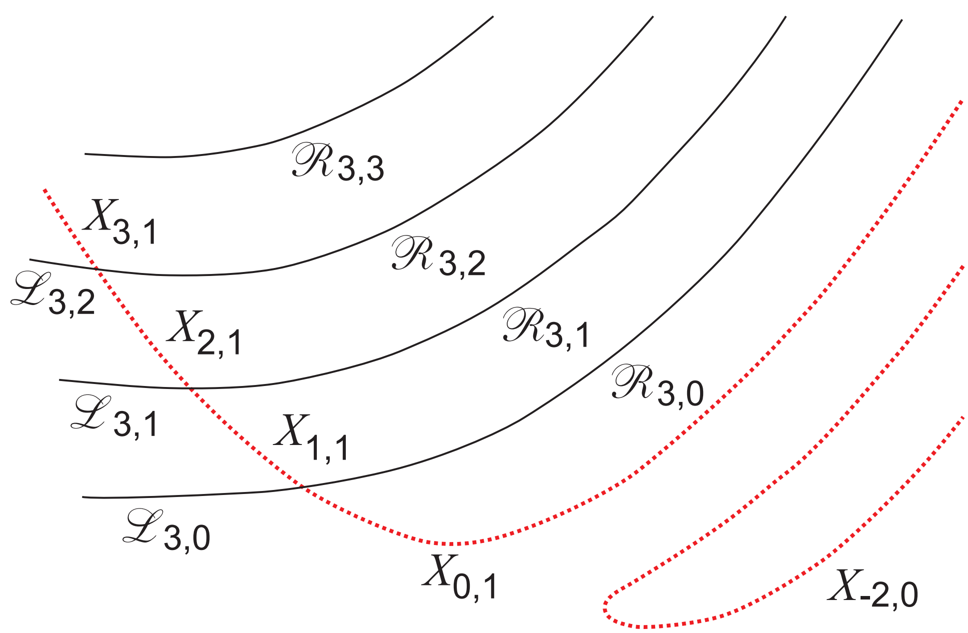

Fig. 6. $Z_3(\mathbf{R})$.

Methods of proofs.

a) Nevanlinna parametrization of the spectral locus.

b) Asymptotics at infinity (singular perturbation theory).

c) Darboux transform of QES quartic.

Nevanlinna parametrization. Let Z be the spectral locus of the problem

$$-y'' + P(z,a)y = \lambda y,\ y(z) \to 0, z \in S_j \cup S_k.$$

Let $(a,\lambda) \in Z$, and y_0 an eigenfunction. Let y_1 be a second linearly independent solution. Then $f = y_0/y_1$ satisfies the Schwarz differential equation

$$\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 = -2(P - \lambda).$$

This function f is meromorphic in $\mathbf{C}$, has no critical points and has $d+2$ asymptotic values, one in each Stokes sector. Asymptotic values in S_j and S_k are 0. Asymptotic values in adjacent sectors are distinct.

In the opposite direction: if we have a meromorphic function in $\mathbf{C}$ without critical points and with finitely many asymptotic values, then it satisfies a Schwarz equation whose RHS is a polynomial. The degree of this polynomial is the number of asymptotic tracts minus 2.

Asymptotic values are meromorphic functions on Z which serve as local parameters. These are Nevanlinna parameters. They are simply related to the Stokes multipliers of the linear ODE.

Functions f of the above type with given set of asymptotic values $A = \{a_0, \ldots, a_{d+1}\}$ have the property that

$$f : \mathbf{C}\backslash f^{-1}(A) \to \overline{\mathbf{C}}\backslash A$$

is a covering map. For a fixed A such covering map can be completely described by certain combinatorial information, a cell decomposition of the plane. These cell decompositions label the charts of our description of the spectral locus.

It is important that we know exactly which cell decompositions can occur and how the cell decomposition changes when the point $(a_0 \ldots, a_{d+1})$ goes over a closed loop in $\mathbf{C}^{d+2}$.

This gives an action of the braid group on the set of special cell decompositions of the plane which can be explicitly computed.

This reduces the problem of parametrization of a spectral locus to combinatorial topology.

For QES operators we use the Darboux transform. Let $-D^2 + V$ be a second order linear differential operator with potential V. Let $\phi_0, \dots \phi_n$ be some eigenfunctions with eigenvalues $\lambda_0, \dots, \lambda_n$. The transformed operator is

$$-D^2 + V - 2\frac{d^2}{dz^2} \log W(\phi_0, \dots, \phi_n),$$

where W is the Wronski determinant. The eigenvalues of the transformed operator are exactly those eigenvalues of $-D^2 + V$ which are *distinct* from $\lambda_0, \dots, \lambda_n$.

We use the Darboux transform to kill the QES part of the spectrum of L_J and it turns out that the transformed operator is L_{-J}!

Our study of the QES locus of the quartic family gives the following interesting identities.

Let h and p be polynomials. When does $y = pe^h$ satisfy a linear differential equation $y'' + Py = 0$ with a polynomial P?

Theorem 10 *TFAE:*

a) $p'' + 2p'h'$ *is divisible by* p,

b) $p^{-2}e^h$ *has no residues,*

c) zeros of p *satisfy the system of equations*

$$\sum_{j:j\neq k} \frac{1}{z_k - z_j} = -h'(z_k),\ 1 \leq k \leq \deg p.$$

Now take $h(z) = z^3/3 - bz$.

Theorem 11 *Let p be a polynomial. All residues of $y = p^{-2}e^{-2h}$ vanish if and only if there exists a constant C and a polynomial q such that*

$$\left(p^2(-z) - \frac{C}{p^2(z)}\right) e^{-2h(z)} = \frac{d}{dz}\left(\frac{q(z)}{p(z)} e^{-2h(z)}\right).$$

Moreover, if this happens then

$$C = (-1)^n 2^{-2n} \frac{\partial}{\partial \lambda} Q_{n+1},$$

where $\lambda = y''/y - z^4 + 2bz^2 - 2(n+1)z$, and $Q_{n+1}(b, \lambda) = 0$ is the equation of the QES spectral locus of L_{n+1}.

This was conjectured in [EG] on the basis of calculations with Darboux transform of L_{n+1} and proved by E. Mukhin and V. Tarasov.

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