SINGULAR PERTURBATION OF POLYNOMIAL POTENTIALS AND REAL SPECTRAL LOCI

Alexandre Eremenko and Andrei Gabrielov

www.math.purdue.edu/~eremenko

www.math.purdue.edu/~agabriel

Purdue University

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0. Brief history

Bender and Wu (1969) studied the even anharmonic oscillator

$$-w'' + (\epsilon x^4 + x^2)w = \lambda w, \quad w(\pm\infty) = 0 \quad (1)$$

as a perturbation of the harmonic oscillator ($\epsilon = 0$). Eigenvalues are analytic functions of $\epsilon > 0$ but have essential singularity at $\epsilon = 0$.

Study of such singularities led Bender and Wu to consideration of *complex potentials* and *boundary conditions in the complex plane*.

Eigenvalues of Problem (1), as functions of complex ϵ, have only algebraic singularities for $\epsilon \neq 0$, while $\epsilon = 0$ is a complicated non-isolated singularity. (Simon (1970), Loeffel and Martin (1972), Delabaere, Dillinger and Pham (1997), Eremenko and Gabrielov (2009). Further we refer to this last paper as EG09.)

1. Eigenvalue problem:

$$-w'' + P(z)w = \lambda w, \qquad (2)$$

$$w(z) \to 0 \text{ as } z \to \infty \text{ on } L_1 \text{ and } L_2 \qquad (3)$$

where $P(z) = a_d z^d + \ldots + a_1 z$, and $L_k = \{re^{i\theta_k},\ r > 0\}$.

Separation rays

$$\text{Re}\left(\int_0^z \sqrt{a_d \zeta^d} d\zeta\right) = 0, \text{ that is } \quad a_d z^{d+2} < 0,$$

divide the plane into $d+2$ sectors $S_0, \ldots, S_{d+1}$. Solution $w \neq 0$ of (2) is *subdominant* in S_j if

$$w(z) \to 0,\ z \to \infty,\ z \in S_j.$$

For each j, the space of subdominant in S_j solutions is 1-dimensional, and no solution can be subdominant in adjacent sectors.

Definition. The rays L_1, L_2 are *admissible* for P if they are not parallel to any separation rays and belong to non-adjacent sectors S_j.

The spectrum of this problem with admissible L_1, L_2 is discrete and infinite. If $a_d = 1$ and $\mathbf{a} = (a_1, \ldots, a_{d-1})$ then there exists an entire function F, called the *spectral determinant*, such that the spectrum is given by the equation

$$F(\mathbf{a}, \lambda) = 0.$$

The set of all solutions of this equation in the $(\mathbf{a}, \lambda)$ space is called the *spectral locus*.

We study global topology of the spectral locus. For example:

For every $d \geq 3$ the spectral locus is a smooth connected hypersurface in $\mathbf{C}^d$. (Alexandersson and Gabrielov (2010); case $d = 3$: EG09).

For $d = 4$, the spectral locus of even potentials P consists of two disjoint smooth connected curves in $\mathbf{C}^2$ (EG09).

2. Self-adjoint and PT-symmetric problems

If P is real and L_1, $L_2 \subset \mathbf{R}$, the problem is self-adjoint and the spectrum is real.

If $P(-\overline{z}) = \overline{P(z)}$, and the rays L_1, L_2 are *interchanged* by the reflection in $i\mathbf{R}$, the problem is called PT-symmetric.* In this case, the spectral determinant is a *real entire function* but some eigenvalues may be non-real.

For the PT-symmetric cubic potential

$$P(z) = iz^3 + iaz \quad \text{and} \quad L_1, L_2 \subset \mathbf{R}, \qquad (4)$$

the spectrum is real if $a \geq 0$ (Case $a = 0$: Dorey, Dunning, Tateo (2001); general case: Shin (2002)).

*P in PT stands for parity and T for time. For mathematics, it does not matter which reflection to consider. It is important that the potential and the boundary conditions are preserved by the symmetry.

The following computer-generated plot of the real spectral locus of (4) is taken from Trinh's thesis (2002). One of our goals is to *prove* that the spectral locus really looks like this.

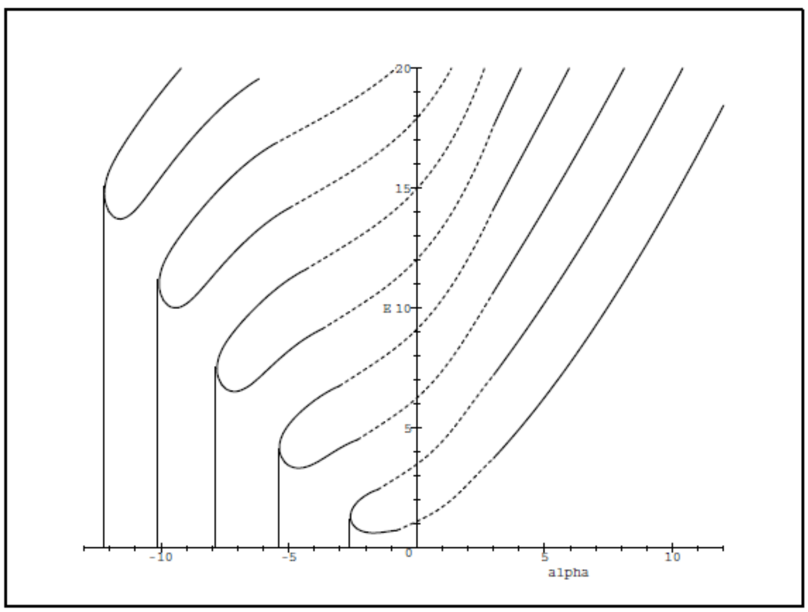

Fig 1. Real spectral locus for PT-symmetric cubic.

Our method is based on "Nevanlinna parametrization" of the spectral locus introduced in EG09, and "degeneration" (singular perturbation) of potentials.

Degeneration results show what happens when $\mathbf{a} \to \infty$ while a_d is fixed. By rescaling, this is equivalent to $a_d \to 0$, while $\mathbf{a}$ is bounded. So we consider potentials

$$P_t(z) = tz^d + cz^m + p_t(z) \tag{5}$$

where $m < d$, $c \in \mathbf{C}\backslash\{0\}$ is fixed, $\deg p_t < m$, coefficients of p_t are bounded, and $t \searrow 0$.

We'll give sufficient conditions for the spectrum of P_t to converge to the spectrum of P_0 as $t \searrow 0$.

First we study the model case $p_t = 0$, and then extend the results to the general case.

3. Stokes complexes of the binomials

The asymptotic behavior of solutions of the equation $-w'' + Pw = \lambda w$ depends on

$$\int \sqrt{P(z) - \lambda}\, dz,$$

which leads to the question of the structure of trajectories of the quadratic differential

$$Q(z)dz^2, \quad Q = P - \lambda.$$

The zeros of Q are called *turning points*. Curves where $Q(z)dz^2 < 0$ are called *vertical trajectories* and curves where $Q(z)dz^2 > 0$ *horizontal trajectories*. Vertical (horizontal) trajectories adjacent to the turning points are called the *Stokes lines* (*anti-Stokes lines*).

Stokes lines and turning points form the 1-skeleton of the cell decomposition of the plane which is called the *Stokes complex*. The 2-cells of this decomposition are called *faces*. The multi-valued function

$$\int \sqrt{Q(z)}dz$$

splits into single-valued branches in the faces. Each branch maps its face onto a right half-plane, or a left half-plane, or onto a vertical strip.

Examples of Stokes complexes

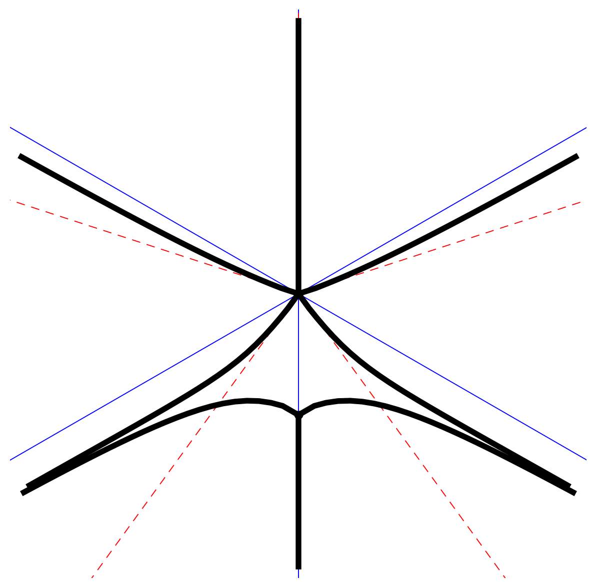

Fig 2. Stokes complex of $z^4 + iz^3$.

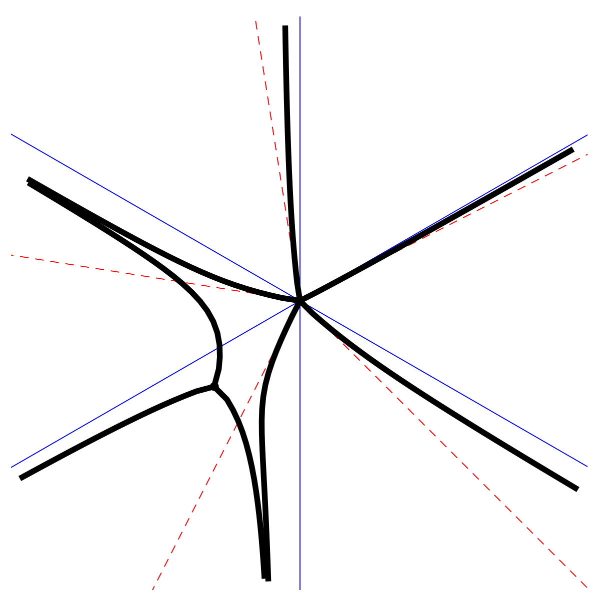

Fig 3. Stokes complex of $z^4 + e^{\pi i/4}z^3$.

Let $Q(z) = z^d + cz^m$. Consider the partition $S(Q)$ of the plane into open sectors and rays defined by the Stokes lines of monomials $z^d\,dz^2$ and $cz^m\,dz^2$ (i.e., by the rays where $z^{d+2} < 0$ or $cz^{m+2} < 0$). Let $R(Q)$ be the refinement of $S(Q)$ by the rays from the origin to non-zero turning points.

Definitions A ray L from the origin is called *good* if it is distinct from the rays of $R(Q)$ and is not tangent to any vertical line of Q.

A sector of $R(Q)$ is good if each ray in it is good.

Theorem 1. Every sector of $R(Q)$ that contains an anti-Stokes line of $z^d\,dz^2$ is good. Every good ray belongs to such a sector.

Theorem 2. *If L_1 and L_2 are good rays for the binomial $z^d + cz^m$, then the discrete spectrum of*

$$P_t = tz^d + cz^m + p_t(z)$$

with the zero boundary conditions on L_1 and L_2 depends continuously on $t \geq 0$.

Convergence of the spectra means that the spectral determinants converge uniformly with respect to λ and coefficients of p_t, when λ and these coefficients are restricted to a compact set.

In other words, for every eigenvalue $\lambda_0 \in K$ for $t = 0$, there exists a unique eigenvalue λ_t which converges to λ_0 as $t \to 0$, and this convergence is uniform with respect to coefficients of p_t restricted to a compact set.

Theorem 2 includes the case when P_0 has no eigenvalues. Then the conclusion means that eigenvalues of P_t escape to infinity as $t \to 0$.

Our proof of Theorem 2 uses a conformal change of the independent variable

$$\zeta = \int \sqrt{P - \lambda}\, dz$$

which is due to Green and Liouville. After this change, the problem is reduced to an integral equation which is solved by successive approximation. Theorem 1 ensures that this change of the variable behaves continuously at $t = 0$, and that the error terms in this successive approximation can be controlled uniformly.

In our applications to spectral loci we need the following Stokes complex:

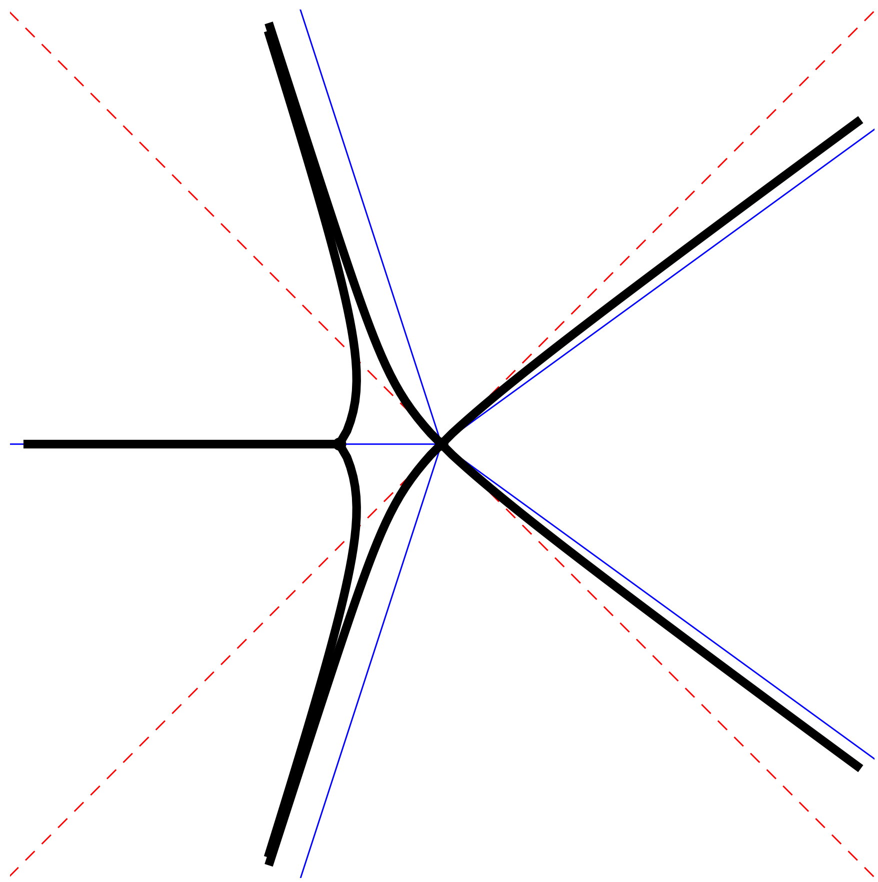

Fig 4. Stokes complex of $z^3 + z^2$. Sectors intersecting the imaginary axis are good.

4. Real spectral locus for the cubic

$$-w'' + (z^3 - az + \lambda) = 0, \quad w(\pm i\infty) = 0. \quad (6)$$

This problem is equivalent to the PT-symmetric problem with

$$P(z) = iz^3 + iaz \quad \text{and} \quad L_1, L_2 \subset \mathbf{R}$$

by the change of the variable $z \mapsto iz$.

Theorem 3. *For every integer $n \geq 0$, there exists a simple curve $\Gamma_n \subset \mathbf{R}^2$, which is the image of a proper analytic embedding of a line, and which has these properties:*
(i) For every $(a, \lambda) \in \Gamma_n$ problem (6) has an eigenfunction with $2n$ non-real zeros.
(ii) The curves Γ_n are disjoint and the real spectral locus of (6) is $\bigcup_{n\geq 0} \Gamma_n$
(iii) The map

$$\Gamma_n \cap \{(a, \lambda) : a \geq 0\} \to \mathbf{R}_{\geq 0},$$

$(a, \lambda) \mapsto a$ is a 2-to-1 covering.
(iv) For $a \geq 0$, $(a, \lambda) \in \Gamma_n$ and $(a, \lambda') \in \Gamma_{n+1}$ imply $\lambda' > \lambda$.

Sketch of the proof.

a) Nevanlinna parametrization of the real spectral locus. Consider the following cell decomposition Φ of the Riemann sphere, with labeled faces. Here $b = e^{i\beta},\ \beta \in (0, \pi)$.

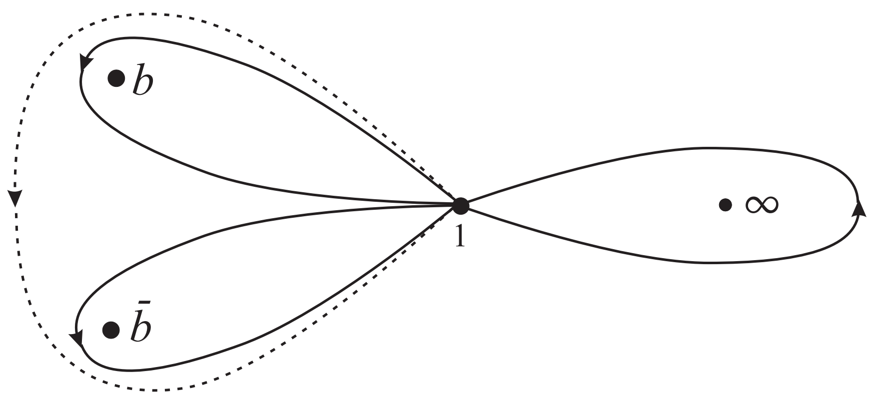

Fig 5. Cell decomposition Φ of the sphere.

and the following cell decompositions Ψ_n of the complex plane:

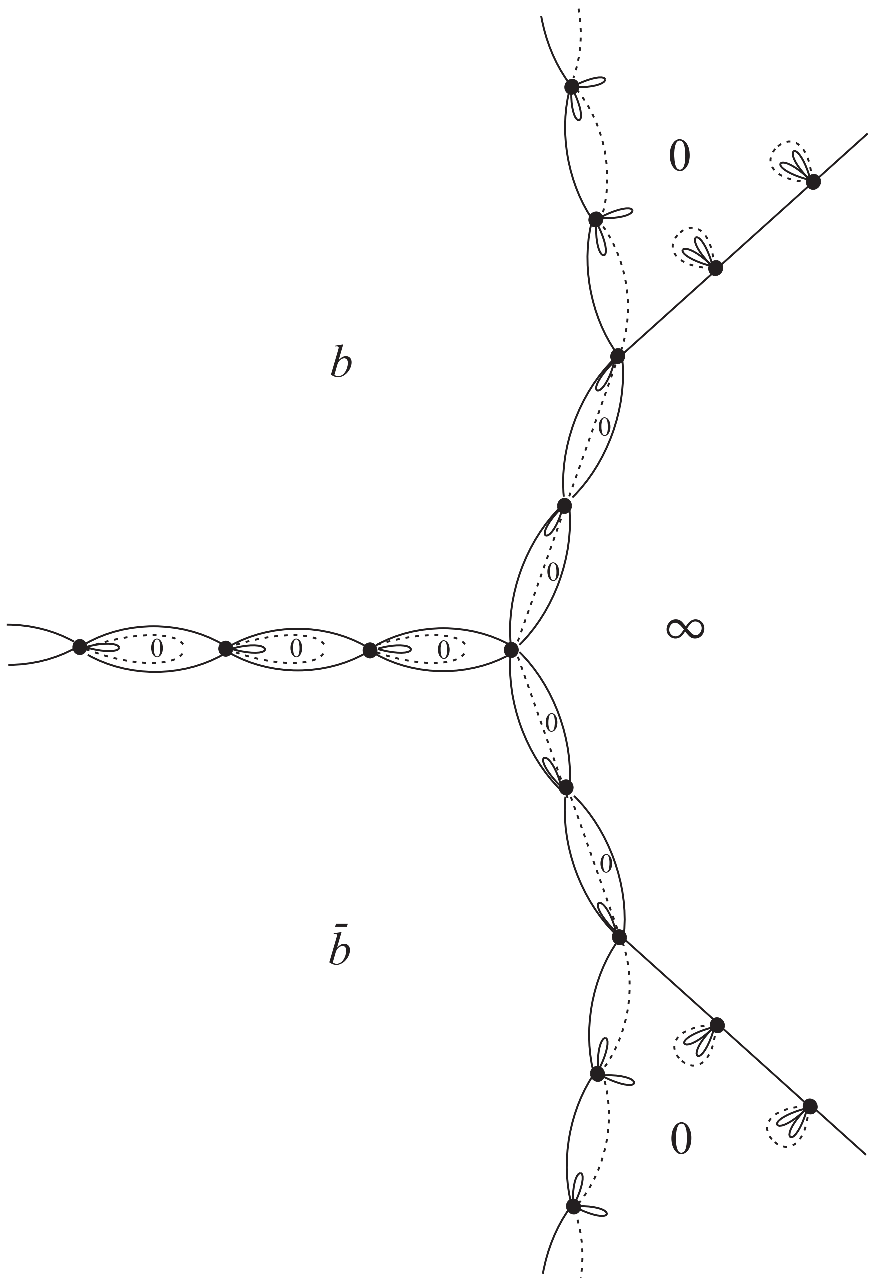

Fig 6. Cell decomposition Ψ_2 of the plane.

Labeled cell decomposition Ψ_n have the same local structure as Φ, so there exists a local homeomorphism $g : \mathbf{C} \to \overline{C}$ such that $\Psi_n = g^{-1}(\Phi)$. This g can be chosen so that it commutes with the reflection $z \mapsto \bar{z}$. By the Uniformization Theorem, there exists a symmetric homeomorphism $\phi : \mathbf{C} \to \mathbf{C}$ such that $f = g \circ \phi$ is a real meromorphic function. According to Nevanlinna theory, this function satisfies the differential equation

$$\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 = -2(z^3 - az + \lambda)$$

with real a, λ. Now we have $f = w/w_1$ where w, w_1 are two real linearly independent solutions of the equation

$$-w'' + (z^3 - az + \lambda)w = 0,$$

and from our construction follows that w is subdominant in the sectors intersecting the imaginary axis. So λ is an eigenvalue. The eigenfunction w has $2n$ non-real zeros, by construction. Thus we have a curve Γ_n in the real spectral locus, parametrized by $b = e^{i\beta}$, $\beta \in (0, \pi)$.

That the union of these curves exhaust the whole real spectral locus follows by reversing the steps, and applying the classification of cell decompositions from EG09a. This proves (i) and (ii). (iii) follows from a result of Shin that for $a > 0$ all eigenvalues are real. To show (iv) we apply rescaling and our previous results on the continuous behavior of the spectrum under degeneration $a \to +\infty$. Real affine change of the independent variable gives an equation $-y'' + (tz^3 + z^2 + \mu)y = 0$, where μ is an explicit increasing function of λ. As $t \searrow 0$ this tends to a self-adjoint problem (harmonic oscillator). The convergence of spectra is justified using Theorems 1 and 2, see Fig. 4. Then the Sturm–Liouville theory gives the relation between the order of eigenvalues and number of zeros of eigenfunctions.

It remains to verify that the number of non-real zeros of an eigenfunction does not change in this degeneration. For this we consider the degeneration of the cell decomposition Ψ_n:

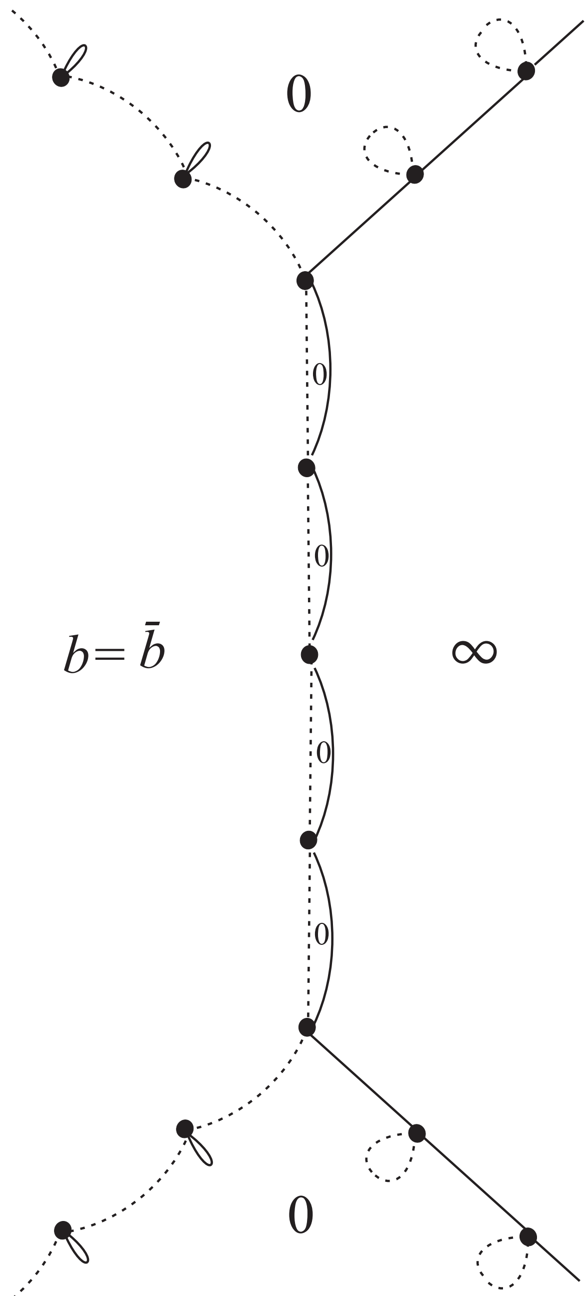

Fig 7. Degenerated cell decomposition of the plane: the loops around b and $\overline{b}$ are replaced by a single loop.

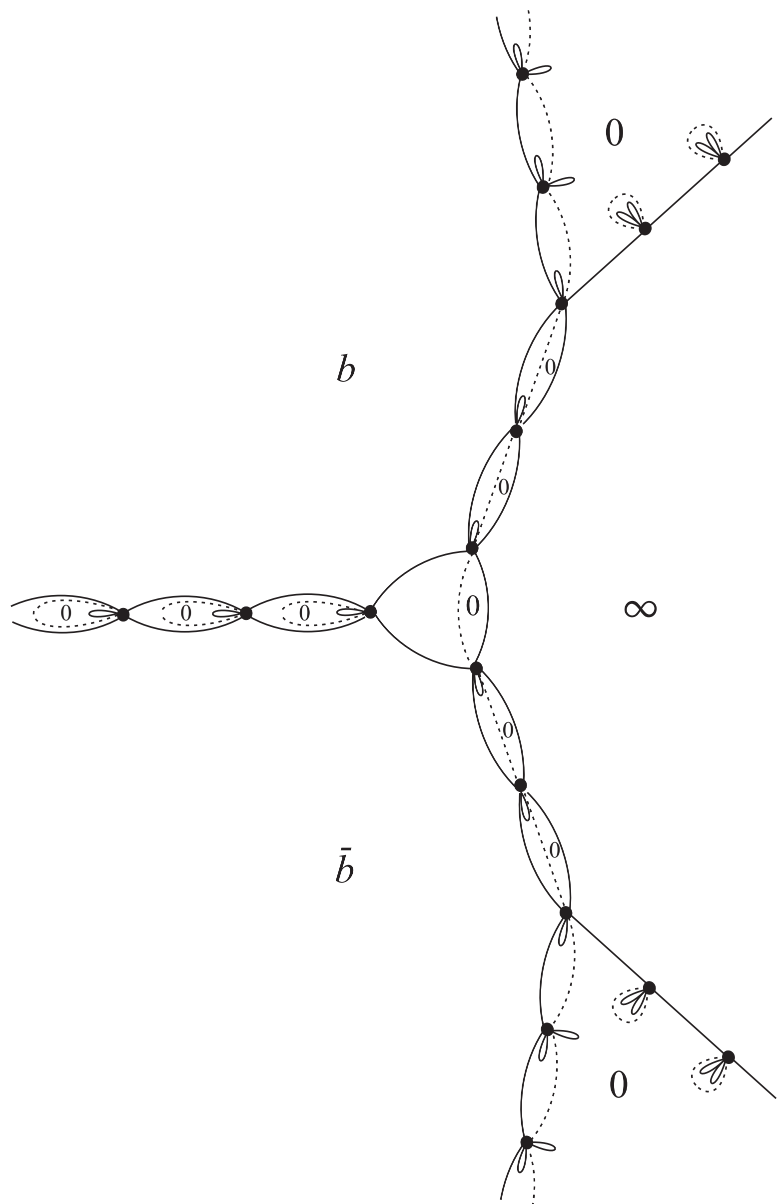

Fig 8. Another cell decomposition for the same cubic.

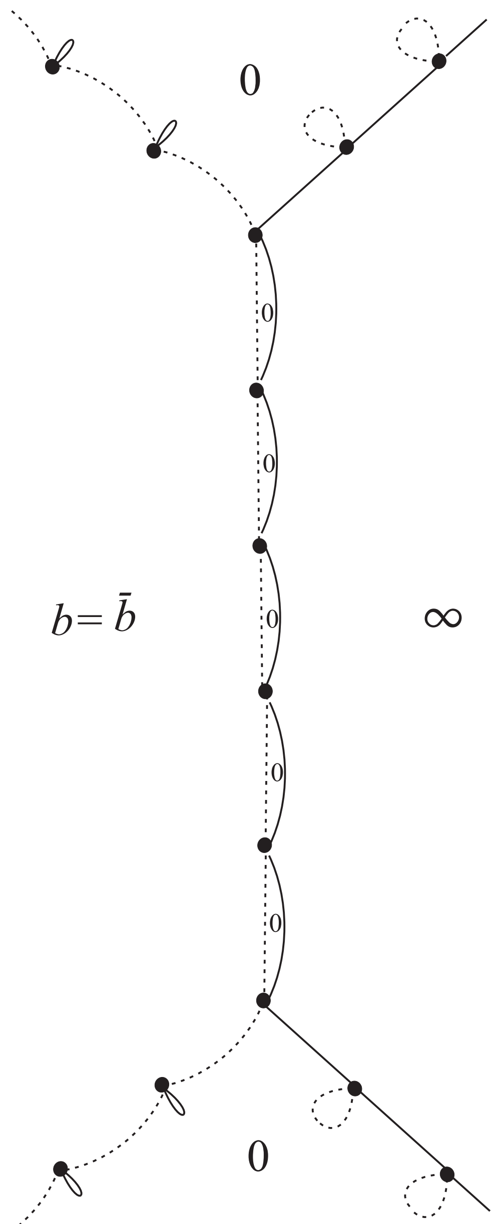

Fig 9. The result of collapse of the second cell decomposition.

5. Real spectral locus for a PT-symmetric quartic family

This is a 2-parametric family

$$-w'' + (-z^4 + az^2 + cz)w = -\lambda w,\ w(\pm i\infty) = 0. \tag{7}$$

It is equivalent to the PT-symmetric family

$$-w'' + (z^4 + az^2 + icz)w = \lambda w, \quad w(\pm\infty) = 0,$$

studied by Bender, *et al* (2001) and Delabaere and Pham (1998).

Theorem 4. *The real spectral locus of (7) consists of disjoint smooth analytic properly embedded surfaces* $S_n \subset \mathbf{R}^3$, $n \geq 0$, *homeomorphic to a punctured disk. For* $(a, c, \lambda) \in S_n$, *the eigenfunction has exactly* $2n$ *non-real zeros. For large* a, *projection of* S_n *on the* (a, c) *plane approximates the region* $9c^2 - 4a^3 \leq 0$.

Numerical computation suggests that the surfaces have the shape of infinite funnels with the sharp end stretching towards $a = -\infty$, $c = 0$, and that the section of S_n by every plane $a = a_0$ is a closed curve.
Theorem 4 implies that this section is compact for large a_0.
The following computer-generated plot is taken from Trinh's thesis:

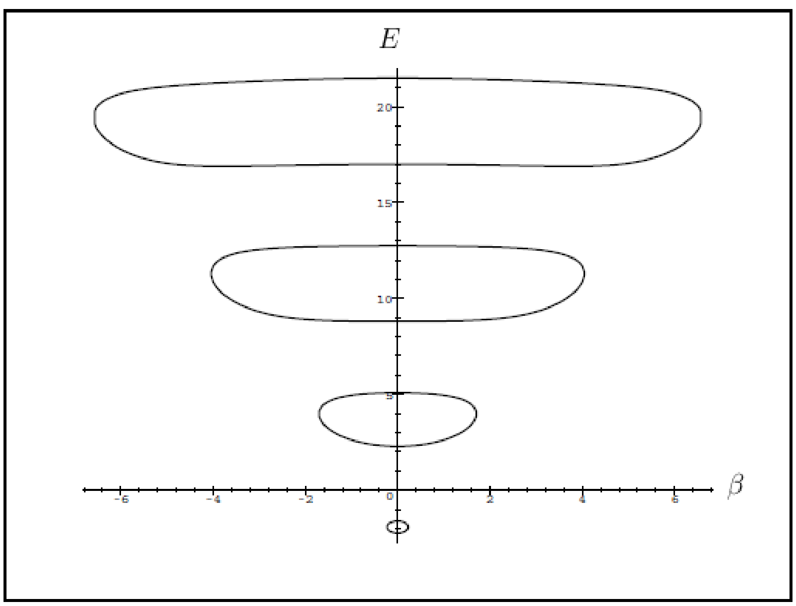

Fig 10. Section of the surfaces $S_0, \ldots, S_3$ by the plane $a = -9$.

Proof of Theorem 4 follows the same lines as the proof of Theorem 3. The cell decompositions now look like this:

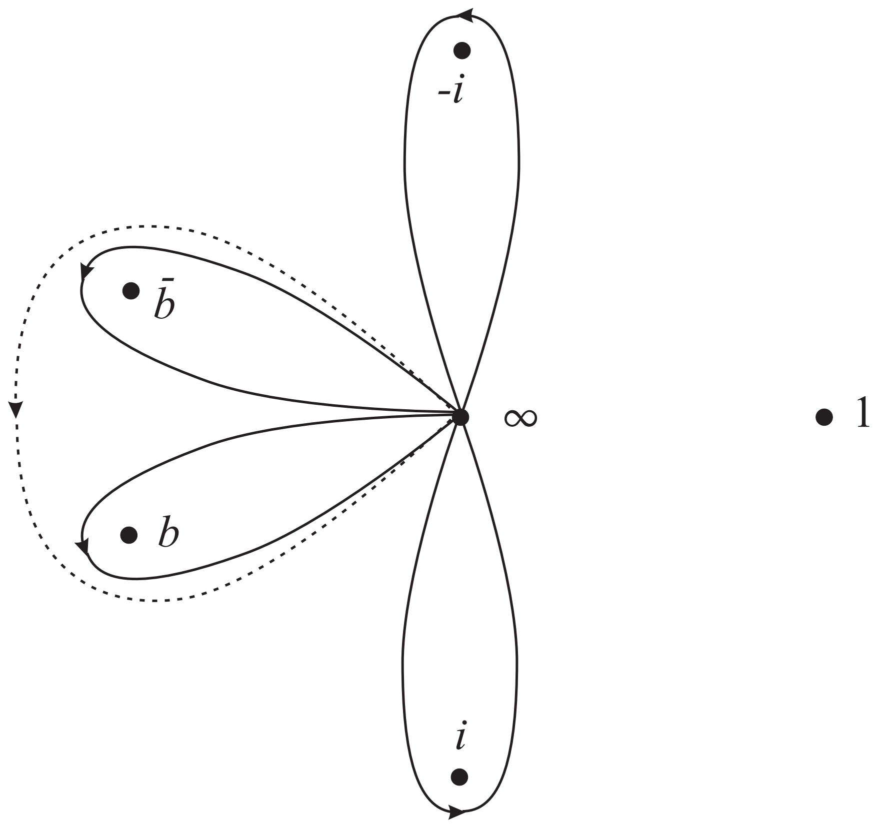

Fig 11. Cell decomposition Φ of the sphere for Theorem 4.

Here the Nevanlinna parameter b belongs to the upper half-plane punctured at i, which explains why S_n is doubly connected.

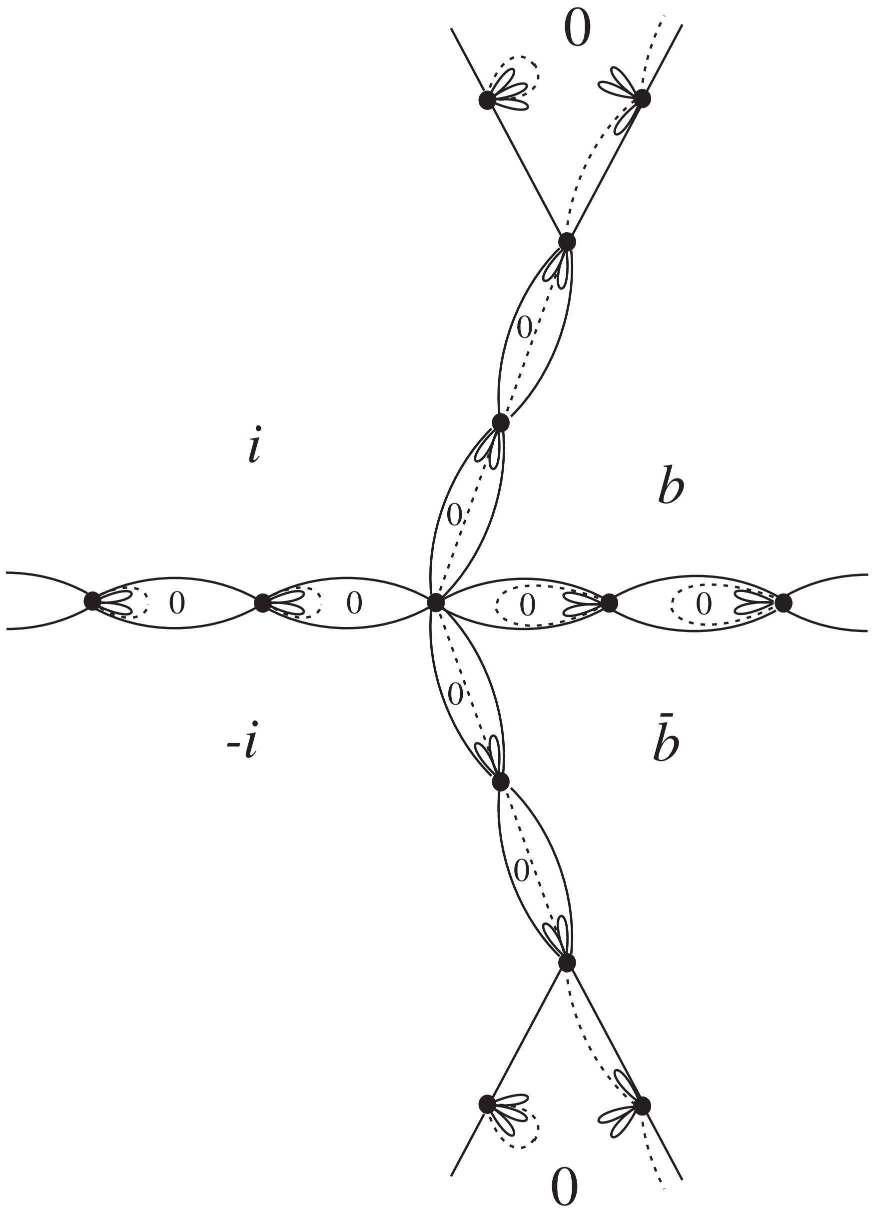

Fig 12. Cell decomposition Ψ_2 of the plane for Theorem 4.

To study asymptotics of S_n as a and c tend to infinity, after an affine change of the independent variable we obtain

$$-y'' + (-tz^4 + z^3 + \alpha z)y = -\mu y.$$

Theorems 1 and 2 imply that the spectrum changes continuously at $t = 0$.

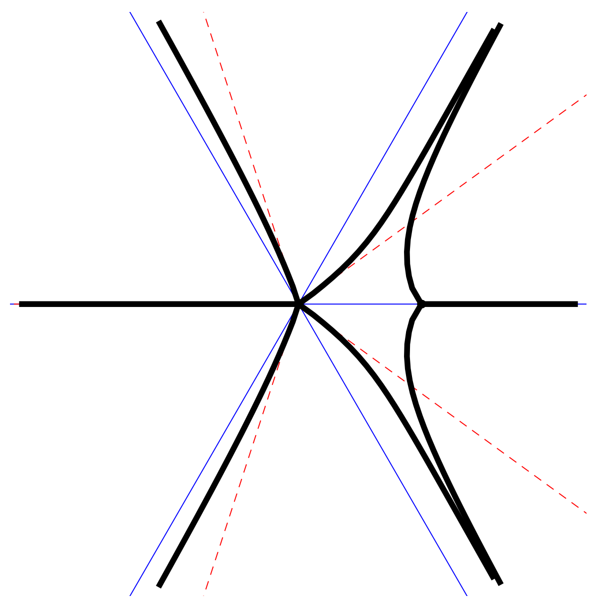

Fig 13. Stokes complex of $-z^4 + z^3$. Sectors intersecting the imaginary axis are good.

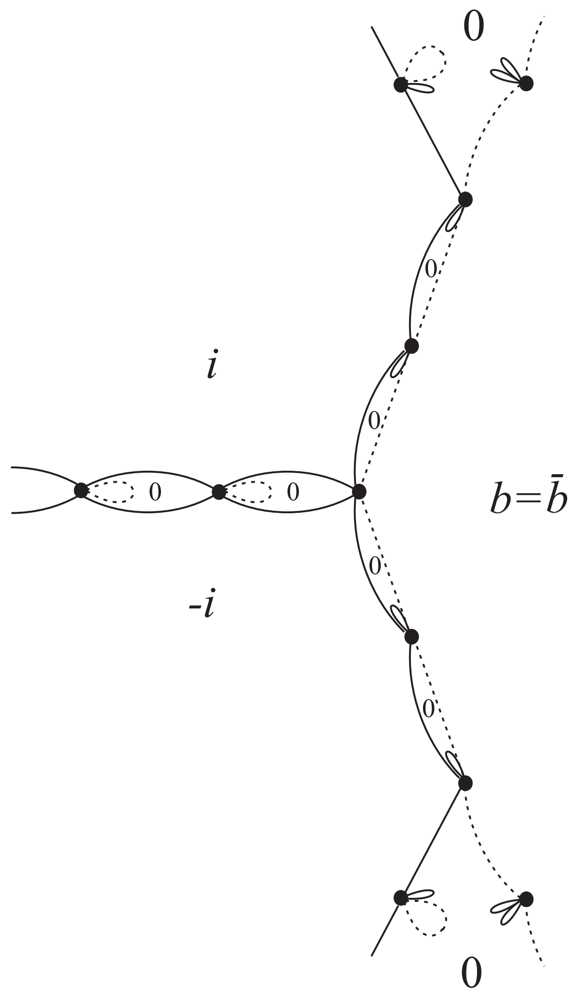

Fig 14. Degenerated cell decomposition for Theorem 4.

The rescaled potential converges to the previously studied cubic potential, and we can make conclusions about asymptotic behavior of the spectral locus.

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