SINGULAR PERTURBATION OF POLYNOMIAL POTENTIALS IN THE COMPLEX DOMAIN WITH APPLICATIONS TO PT-SYMMETRIC FAMILIES

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Abstract. We discuss eigenvalue problems of the form $-w'' + Pw = \lambda w$ with complex polynomial potential $P(z) = tz^d + \ldots$, where t is a parameter, with zero boundary conditions at infinity on two rays in the complex plane. In the first part of the paper we give sufficient conditions for continuity of the spectrum at $t = 0$. In the second part we apply these results to the study of topology and geometry of the real spectral loci of PT-symmetric families with P of degree 3 and 4, and prove several related results on the location of zeros of their eigenfunctions.

1. Introduction

We consider eigenvalue problems

$$(1.1)\qquad -w'' + P_{\mathbf{a}}(z)w = \lambda w, \quad y(z) \to 0 \text{ as } z \to \infty,\ z \in L_1 \cup L_2.$$

Here P is a polynomial in the independent variable z, which depends on a parameter $\mathbf{a}$, and L_1, L_2 are two rays in the complex plane. The set of all pairs $(\mathbf{a}, \lambda)$ such that λ is an eigenvalue of (1.1) is called the *spectral locus*.

Such problems where considered for the first time in full generality by Sibuya [38] and Bakken [2]. Sibuya proved that under certain conditions on L_1, L_2 and the leading coefficient of P, there exists an infinite sequence of eigenvalues tending to infinity. These eigenvalues are roots of an entire function which is called the *spectral determinant*. If

$$(1.2)\qquad P_{\mathbf{a}}(z) = z^d + a_{d-1}z^{d-1} + \ldots + a_1 z,$$

then the spectral determinant is an entire function of d variables $F(\mathbf{a}, \lambda)$, where $\mathbf{a} = (a_1, \ldots, a_{d-1})$, such that the spectral locus, which is the set of all $(\mathbf{a}, \lambda) \in \mathbf{C}^d$ where λ is an eigenvalue of (1.1), is described by the equation $F(\mathbf{a}, \lambda) = 0$. So the spectral locus of (1.1), (1.2) is an analytic hypersurface in $\mathbf{C}^d$. It is smooth [2] and connected for $d \geq 3$, [1], [20].

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In the first part of this paper we study what happens to the eigenvalues and eigenfunctions when the leading coefficient of P tends to zero. Related question was investigated by Sibuya in [38, Ch. 3]. He considered the behavior of certain normalized solutions of the equation (1.1) with $P(z) = tz^d + a_{d-1}z^{d-1} + \ldots + a_1 z$ in the case when both t and a_{d-1} are positive, $t \to 0+$, while a_{d-1} is fixed, and the normalization ray is the positive ray. He did not state any conclusions about the behavior of eigenvalues.

Bender and Wu [8] studied the quartic oscillator as a perturbation of the harmonic oscillator:

$$-w'' + (\varepsilon z^4 + z^2)w = \lambda w, \quad w(\pm\infty) = 0. \tag{1.3}$$

Here and in what follows $w(\pm\infty) = 0$ means that the boundary conditions are imposed on the positive and negative rays of the real line. It has been known for long time that the eigenvalues of (1.3) converge as $\varepsilon \to 0+$ to the eigenvalues of the same problem with $\varepsilon = 0$, but they are not analytic functions of ε at $\varepsilon = 0$ (perturbation series diverge). To investigate this phenomenon, Bender and Wu considered complex values of ε and studied the analytic continuation of the eigenvalues as functions of ε in the complex plane. Their main findings can be stated as follows: the spectral locus of the problem (1.3) consists of exactly two connected components; for $\varepsilon \neq 0$, the only singularities of eigenvalues as functions of ε are algebraic branch points. These statements were rigorously proved in [19]. Discoveries of Bender and Wu generated large literature in physics and mathematics. For a comprehensive exposition of the early rigorous results we refer to [39].

To perform analytic continuation of eigenvalues of (1.3) and similar problems for complex parameters, one has to move the normalization rays where the boundary conditions are imposed. One of the early papers in the physics literature that emphasized this point was [7]. Thus physicists were led to problem (3.1), previously studied for its intrinsic mathematical interest.

An interesting phenomenon was discovered by Bessis and Zinn-Justin. For the boundary value problem

$$-w'' + iz^3 w = \lambda w, \quad w(\pm\infty) = 0,$$

they found by numerical computation that the spectrum is real. This is called the Bessis and Zinn-Justin conjecture (see, for example, historical remark in [5]). This conjecture was later proved by Dorey, Dunning and Tateo [15, 16] with a remarkable argument which they call the ODE-IM correspondence, see their survey [17]. The functional equation on which this argument is based can be found in Sibuya's book [38]. Shin [35] extended this result to potentials

$$-w'' + (iz^3 + iaz)w = \lambda w, \quad w(\pm\infty) = 0. \tag{1.4}$$

These results and conjectures generated extensive research on the so-called PT-symmetric boundary value problems. PT-symmetry means a symmetry of the potential and of the boundary conditions with respect to the reflection in the imaginary line $z \mapsto -\overline{z}$.

PT stands for "parity and time reversal". From pure mathematical point of view it does not matter which line is chosen for the reflection symmetry. The essential difference between a PT-symmetric and an Hermitian boundary value problem is that PT symmetry interchanges the two boundary conditions while Hermitian symmetry leaves each of them fixed.

It turns out that the spectral determinant of a PT-symmetric problem is a real entire function of λ, so the spectrum is invariant under complex conjugation. In contrast to Hermitian problems where the spectrum is always real, the spectrum of a PT-symmetric problem can be real for some values of parameters, but for other values of parameters some eigenvalues may be complex. So we can see the "level crossing" (collision of real eigenvalues) in real analytic families of PT-symmetric operators, the phenomenon which is impossible in the families of Hermitian differential operators with polynomial coefficients.

In this paper, we first consider the general problem (1.1) and the limit behavior of its eigenvalues and eigenfunctions when $P(z) = tz^d + a_m z^m + p(z)$, with $d > m > \deg p$, as $t \to 0$, while the coefficients of p are restricted to a compact set and a_m has a non-zero limit. Then we apply our general results to certain families of PT-symmetric potentials of degrees 3 and 4, and prove some conjectures made by several authors on the basis of numerical evidence.

In particular, our results for the PT-symmetric cubic (1.4) imply that no eigenvalue can be analytically continued along the negative a-axis, and the obstacle to this continuation is a branch point where eigenvalues collide.

Another result is the correspondence between the natural ordering of real eigenvalues of (1.4) for $a \geq 0$ and the number of zeros of eigenfunctions that do not lie on the PT-symmetry axis, conjectured by Trinh in [42]. This correspondence is similar to that given by the Sturm–Liouville theory for Hermitian boundary value problems.

A different approach to counting zeros of eigenfunctions is proposed in [26], where the authors prove that for a large enough, the n-th eigenfunction has n zeros in certain explicitly described region in the complex plane.

The plan of the paper is the following. In Section 2 we first study Stokes complexes of binomial potentials $P(z) = tz^d + a_m z^m,\ d > m$. Stokes complex is the union of curves, passing through the zeros of $P - \lambda$, where the quadratic differential $(P(z) - \lambda)\, dz^2$ is negative. It occurs in all questions about asymptotic behavior of solutions of equations (1.1). This study permits us to make conclusions on the behavior, as $t \to 0$, of the Stokes complexes of potentials $P(z) = tz^d + a_m(t)z^m + p_t(z)$ where $a_m(t)$ tends to a non-zero limit and p_t is a family of polynomials of degree $m-1$ with bounded coefficients. On our opinion, investigation of binomial Stokes complexes is of some intrinsic interest, independent of this application. We mention [33] where a topological classification of Stokes complexes for polynomials of degree 3 is given. This study suggests that the spectrum changes continuously as $t \to 0$ when there is certain relation between the arguments of t, a_m and the rays L_1, L_2. This we prove in Section 3, namely that under some conditions on the rays L_1, L_2, and the arguments of t, a_m, the eigenvalues and eigenfunctions for $t = 0$ are the limits of the eigenvalues and eigenfunctions as $t \to 0$, assuming that a_m has non-zero limit.

In the rest of the paper we apply these results to concrete problems. In Section 4, we consider the PT-symmetric cubic family (1.4) with real a and λ. We prove that the intersection of the spectral locus with the real (a, λ)-plane consists of disjoint non-singular analytic curves $\Gamma_n,\ n \geq 0$, the fact previously known from numerical computation [14, 41, 29]. Moreover, we prove that the eigenfunctions corresponding to $(a, \lambda) \in \Gamma_n$ have exactly $2n$ zeros outside of the imaginary line. (They have infinitely many zeros on the imaginary line). Furthermore, using the result of Shin on reality of eigenvalues, we study the shape and relative location of these curves Γ_n in the (a, λ)-plane and show that $a \to +\infty$ on both ends of Γ_n, and that for $a \geq 0$, Γ_n consists of graphs of two functions, that lie below the graphs of functions constituting Γ_{n+1}.

This gives PT–analog of the familiar fact for Hermitian boundary value problems that "n-th eigenfunction has n real zeros"; in our case we count zeros belonging to a certain well-defined set in the complex plane. This result proves rigorously what can be seen in numerical computations of zeros of eigenfunctions by Bender, Boettcher and Savage [6].

The result of Section 4 also gives a contribution to a problem raised by Hellerstein and Rossi [9]: describe the differential equations

$$y'' + Py = 0 \tag{1.5}$$

with polynomial coefficient P which have a solution whose all zeros are real. For polynomials of degree 3, all such equations are parametrized by our curve Γ_0, and equations having solutions with exactly $2n$ non-real zeros are parametrized by Γ_n.

The arguments in Section 4 use our parametrization of the spectral loci from [21, 19] combined with the singular perturbation results of Sections 2 and 3. These perturbation results allow us to degenerate the cubic potential to a quadratic one (harmonic oscillator) and to make topological conclusions based on the ordinary Sturm-Liouville theory.

Next we apply similar methods to two families of PT-symmetric quartics

$$-w'' + (z^4 + az^2 + icz)w = \lambda y, \quad w(\pm\infty) = 0. \tag{1.6}$$

and

$$w'' + (z^4 + 2az^2 + 2imz + \lambda)y = 0, \tag{1.7}$$

$$w(re^{\theta}) \to 0, \text{ as } r \to \infty, \; \theta \in \{-\pi/6, -5\pi/6\}, \tag{1.8}$$

where $m \geq 1$ is an integer. The first family was considered in [3] and [12, 13]. We prove that the spectral locus in the real (a, c, λ)-space $\mathbf{R}^3$ consists of infinitely many smooth analytic surfaces S_n, $n \geq 0$, each homeomorphic to a punctured disc, and that an eigenfunction corresponding to a point $(a, c, \lambda) \in S_n$ has exactly $2n$ zeros which do not lie on the imaginary axis. We study the shape and position of these surfaces by degenerating the quartic potential to the previously studied PT-symmetric cubic oscillator.

The second quartic family (1.7-1.8) was introduced by Bender and Boettcher [4]. It is quasi-exactly solvable (QES) in the sense that for every integer $m \geq 1$ in the potential, there are m "elementary" eigenfunctions, each having $m - 1$ zeros. The part Z_m of the spectral locus corresponding to these elementary eigenfunctions is a smooth connected curve in $\mathbf{C}^2$ [19, 20]. In the end of Section 4 we study the intersection of this curve with the real (a, λ)-plane. Similarly to the case of the PT-symmetric cubic, this intersection consists of smooth analytic curves $\Gamma^*_{m,n}$, $n = 0, \ldots, \lceil m/2 \rceil$, and for $(a, \lambda) \in \Gamma^*_{m,n}$ the eigenfunction has exactly $2n$ zeros that do not lie on the imaginary axis. For $n \leq m/2$ the part of $\Gamma^*_{m,n}$ over some ray $a > a_m$ consists of disjoint graphs of two functions, and we have the following ordering: $(a, \lambda) \in \Gamma^*_{m,n}$, $(a, \lambda') \in \Gamma^*_{m,n+1}$ and $a > a_m$ imply that $\lambda' > \lambda$. Moreover, the QES spectrum for $a > a_m$ consists of the m smallest real eigenvalues.

The results of Section 4 permit us to answer the question of Hellerstein and Rossi stated above for polynomial potentials of degree 4: All equations (1.5) that possess a solution with $2n$ non-real zeros are parametrized by our curves $\Gamma^*_{m,n}$ if the total number of zeros is $m-1$, and by our surfaces S_n if the total number of zeros is infinite.

Notation and conventions.

1. What we call Stokes lines is called by some authors "anti-Stokes lines" and vise versa. We follow terminology of Evgrafov and Fedoryuk [23, 24].

2. We prefer to replace z by iz in PT-symmetric problems. Then potentials become real, and the difference between PT-symmetric and self-adjoint problems is that in PT-symmetric problems the complex conjugation *interchanges* the two boundary conditions, while in self-adjoint problems both boundary conditions remain fixed by the symmetry. The main advantage for us in this change of the variable is linguistic: we frequently refer to "non-real" zeros. The expression "non-real" excludes 0, while the expression "non-imaginary" does not.

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2. Stokes complexes of binomials

In this section we study Stokes complexes of quadratic differentials $Q(z)\,dz^2$ where Q is a polynomial. For $Q(z) = tz^d + cz^m + q(z)$, with $\deg q \leq m < d,\ t \geq 0$, we give a simple sufficient condition in terms of d, m, c for the Stokes complex to be disjoint from certain sectors (Definition 2.6 and Theorem 2.21). Only this theorem and a technical result in Lemma 2.19 will be used in the rest of the paper.

We begin with recalling terminology. Let $Q(z)$ be a polynomial, $z \in \mathbf{C}$. A *vertical line* of $Q(z)\,dz^2$ is an integral curve of the direction field $Q(z)\,dz^2 < 0$. A *Stokes line* of Q is a vertical line with one or both ends in the set of *turning points* $\{z : Q(z) = 0\}$. The *Stokes complex* of Q is the union of the Stokes lines and turning points. A *horizontal line* of Q is a vertical line of $-Q$. Vertical and horizontal lines intersect orthogonally. An *anti-Stokes line* of Q is a Stokes line of $-Q$.

Every Stokes line has one end at a turning point and the other end either at a different turning point or at infinity. If Q has a zero at z_0 of multiplicity m then there are $m+2$ Stokes lines with the endpoint at z_0; they partition a neighborhood of z_0 into sectors of equal opening $2\pi/(m+2)$. The $m+2$ anti-Stokes lines having one end at z_0 bisect these sectors.

Let $L(\alpha) = \{z \in \mathbf{C} \setminus \{0\} : \arg z = \alpha\}$ and, for $0 < \beta - \alpha < 2\pi$, $S(\alpha, \beta) = \{z \in \mathbf{C} \setminus \{0\} : \alpha < \arg z < \beta\}$. For $R \geq 0$, let $D(R) = \{z \in$

$\mathbf{C} : |z| \leq R\}$. For a ray L or a sector S, define $L_R = L \setminus D(R)$, $S_R = S \setminus D(R)$. For a set $S \subset \mathbf{C}$, $\overline{S}$ is its closure in $\mathbf{C}$ and $\partial S = \overline{S} \setminus S$.

Definition 2.1. A set $S = S_R(\alpha, \beta)$ is called a *trap* for Q if every vertical line of Q inside S having one end on ∂S has its other end at infinity. For $\eta > 0$, a trap S is called an *η-trap* for Q if the vertical lines of Q inside S and the rays through the origin intersect under the angles at most $\pi/2 - \eta$.

Lemma 2.2. *Let $S = S_R(\alpha, \beta)$ satisfy*

(i) $\mathrm{Im}(Q(z)\, dz^2) \geq 0$ *on* $L_R(\alpha)$,
(ii) $\mathrm{Im}(Q(z)\, dz^2) \leq 0$ *on* $L_R(\beta)$,
(iii) *Q does not have non-zero turning points in $\overline{S}$,*
(iv) *horizontal lines of Q are not tangent to $L_R(\theta)$ for all $\theta \in [\alpha, \beta]$.*

Then S is a trap for Q.

Proof. Condition (iv) guarantees that the form dr does not vanish on any vertical line Γ of Q in $S_R(\alpha, \beta)$. Hence Γ can be oriented so that r is increasing on Γ. In particular, Γ cannot exit $L_R(\alpha, \beta)$ through $\{r = R\}$. Condition (iii) guarantees that Γ cannot end at a turning point inside $S_R(\alpha, \beta)$. Conditions (i) and (ii) imply that Γ cannot exit $S_R(\alpha, \beta)$ through either $L_R(\alpha)$ or $L_R(\beta)$. □

Corollary 2.3. *Let $Q(0) = 0$ and $S = S(\alpha, \beta)$ satisfies conditions of Lemma 2.2 with $R = 0$. Then either there is exactly one Stokes line of Q in S with one end at 0 and the other end at infinity, or one of the rays $L(\alpha)$ and $L(\beta)$ is a Stokes line of Q. In the latter case, there are no Stokes lines with the end at 0 in S.*

Proof. Let m be the order of a zero of Q at 0. If neither $L(\alpha)$ nor $L(\beta)$ is a Stokes line of Q, then the union $V(\alpha)$ of vertical lines of Q in S with an end on $L(\alpha)$ is a non-empty open subset of S. The same is true for the corresponding set $V(\beta)$. Since $V(\alpha) \cap V(\beta) = \emptyset$, the complement in S of $V(\alpha) \cup V(\beta)$ is non-empty and consists of vertical lines of Q with an end at 0. Condition (iv) of Lemma 2.2 implies that $\beta \leq \alpha + \pi/(m+2)$, hence there is at most one such line in S. If, e.g., $L(\alpha)$ is a Stokes line of Q then, since $\beta \leq \alpha + \pi/(m+2)$, there are no Stokes lines of Q with an end at 0 in S. Also, $L(\beta)$ cannot be a Stokes line of Q. The case when $L(\beta)$ is a Stokes line of Q is treated similarly. □

Lemma 2.4. *Let $S = S(\alpha, \beta)$ and $Q(z)$ a polynomial with a simple zero at $z_0 \in L(\alpha)$, such that*

(i) $\mathrm{Im}(Q(z)\, dz^2) \geq 0$ *on* $L(\alpha)$,

(ii) $\text{Im}(Q(z)\,dz^2) \leq 0$ *on* $L(\beta)$,
(iii) *the only turning points of* Q *in* $\overline{S_R(\alpha,\beta)}$ *are* 0 *and* z_0,
(iv) *horizontal lines of* Q *are not tangent to* $L_R(\theta)$ *for all* $\theta \in (\alpha,\beta]$.
(v) *the interval* $(0, z_0)$ *of* $L(\alpha)$ *is a Stokes line of* Q.

Then S *is a trap for* Q.

Proof. The same arguments as in the proof of Lemma 2.2 imply that the Stokes line Γ of Q in S with one end at z_0 cannot exit S through $L(\beta)$. If Γ would exit S through $L(\alpha)$, there would be an open domain G in S bounded by Γ and a segment I of the ray $L(\alpha)$ between the two ends of Γ. Every vertical line γ of Q in G must have both ends on I. Condition (iv) implies that the vertical direction of Q must rotate by π on the closed path consisting of γ and a segment of $L(\alpha)$ connecting its ends. This is impossible since G is simply connected and does not contain turning points of Q.

Hence Γ has another end at infinity. Any vertical line γ of Q entering S either through $L(\alpha) \setminus [0, z_0]$ or through $L(\beta)$ cannot intersect Γ and cannot exit through the same ray it entered (due to the same arguments as in Lemma 2.2). Hence γ has its other end at infinity. □

Lemma 2.5. *Let* $S = S(\alpha,\beta)$, *and let* $Q(z)$ *be a polynomial with a simple zero at* $z_0 \in L(\beta)$, *and such that*

(i) $\text{Im}(Q(z)\,dz^2) \geq 0$ *on* $L(\alpha)$,
(ii) $\text{Im}(Q(z)\,dz^2) \leq 0$ *on* $L(\beta)$,
(iii) $(Q(z)\,dz^2) \notin \mathbf{R}_+$ *on* $L(\theta)$ *for each* $\theta \in [\alpha,\beta)$.
(iii) *the only turning points of* Q *in* $\overline{S}$ *are* 0 *and* z_0,
(iv) *horizontal lines of* Q *are not tangent to* $L_R(\theta)$ *for all* $\theta \in [\alpha,\beta)$.
(v) *the interval* $(0, z_0)$ *of* $L(\beta)$ *is a horizontal line of* Q.

Then S *is a trap for* Q.

The proof is the same as that of Lemma 2.4, exchanging α and β.

Definition 2.6. Let $Q(z) = P_d + P_m$ be a binomial, where $P_d = tz^d$ and $P_m = cz^m$ are two non-zero monomials, $0 < m < d$. Let $\mathcal{S}(Q)$ be the partition of $\mathbf{C} \setminus \{0\}$ into sectors and rays defined by the Stokes lines of the two monomials P_d and P_m. Let $\mathcal{T}(Q)$ be the partition of $\mathbf{C} \setminus \{0\}$ into sectors and rays defined by the Stokes and anti-Stokes lines of P_d and P_m. A sector S of the partition $\mathcal{S}(Q)$ is called *stable* if

(a) S contains an anti-Stokes line of P_m.
(b) The closures of the two sectors of the partition $\mathcal{S}(Q)$ adjacent to S do not contain non-zero turning points.

A sector S of $\mathcal{S}(Q)$ is called *marginally stable* if, instead of (b), the following weaker property holds:

(b′) There are no turning points of Q inside the two sectors of the partition $\mathcal{S}(Q)$ adjacent to S.

Lemma 2.7. *A stable or marginally stable sector S contains an anti-Stokes line of P_d.*

Proof. A sector S of $\mathcal{S}(Q)$ cannot be bounded by two Stokes lines of P_m. If a sector S is bounded by two Stokes lines of P_d, its bisector is an anti-Stokes line of P_d. If a stable sector $S = S(\alpha, \beta)$ is bounded by a Stokes line of P_m and a Stokes line of P_d, condition (a) implies that $\beta - \alpha > \pi/(m+2) > \pi/(d+2)$. Since one of the boundaries of S is a Stokes line of P_d, there should be an anti-Stokes line of P_d in S. □

Example 2.8. Fig. 1 shows partitions $\mathcal{T}(Q)$ for $Q = z^4 + iz^3$ and $Q = z^4 + e^{i\pi/4}z^3$. Black solid rays are the Stokes lines of the quartic monomial z^4. Red dashed rays are the Stokes lines of the cubic monomial. Dotted rays are the anti-Stokes lines of the monomials of Q. Black dots are the turning points of Q.

Sectors S_1-S_4 in Fig. 1a and sectors S_0-S_3 in Fig. 1b are stable. Sector S_4 in Fig. 1b is not stable because the sector S_5 adjacent to it contains a turning point.

Fig. 2 shows partitions $\mathcal{T}(Q)$ for $Q = z^4 + z^3$ and $Q = z^3 - z^2$. Notations are the same as in Fig. 1.

Sectors S_0-S_2 in Fig. 2a and sectors S_2-S_3 in Fig. 2b are stable. Sectors S_3-S_4 in Fig. 2a and sectors S_0-S_1 in Fig. 2b are marginally stable.

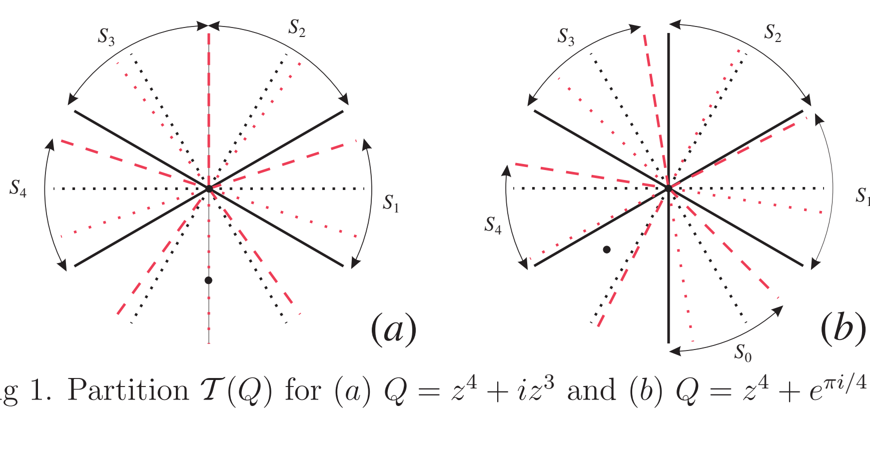

Fig 1. Partition $\mathcal{T}(Q)$ for (a) $Q = z^4 + iz^3$ and (b) $Q = z^4 + e^{\pi i/4}z^3$.

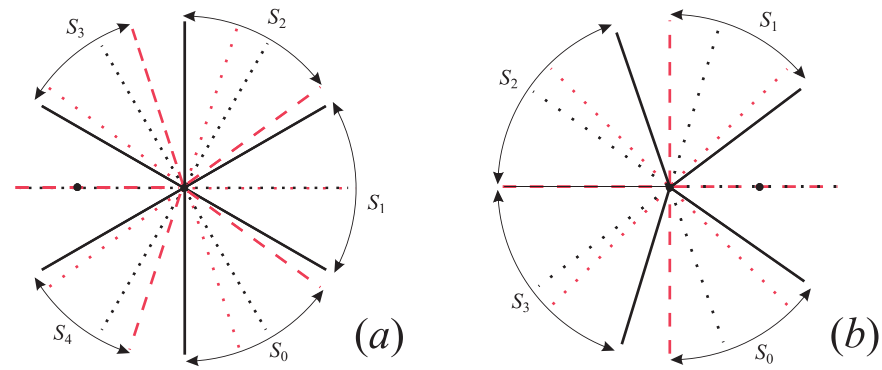

Fig 2. Partition $\mathcal{T}(Q)$ for (a) $Q = z^4 + z^3$ and (b) $Q = z^3 - z^2$.

Lemma 2.9. *Let $Q(z)$ be as in Definition 2.6. Let $S = S(\alpha, \beta)$ be either a sector of the partition $\mathcal{S}(Q)$ that does not contain anti-Stokes lines of P_d (i.e., S is also a sector of the partition $\mathcal{T}(Q)$) or a common Stokes line of P_d and P_m. Let T be a sector containing S and such that T does not contain any turning points of Q, and does not contain any sectors of the partition $\mathcal{T}(Q)$ other than S.*

(a) *If $L(\alpha)$ (resp. $L(\beta)$) contains a turning point of Q then T satisfies conditions of Lemma 2.4 (resp. Lemma 2.5). In particular, T is a trap for Q.*
(b) *If $\overline{S}$ contains no non-zero turning points of Q then T satisfies conditions of Lemma 2.2 with $R = 0$ and Corollary 2.3. Moreover, there exists $\eta > 0$ such that T is an η-trap for Q.*

Proof. Conditions imply that $\beta - \alpha \leq \pi/(d + 2)$. In particular, one of the rays $L(\alpha)$ and $L(\beta)$ is a Stokes line of P_m, and another one a Stokes line of P_d. Suppose that L_α is a Stokes line of P_m. Then $L(\alpha)$ contains a turning point of Q if and only if $\beta = \alpha + \pi/(d + 2)$.

Suppose first that $L(\alpha)$ contains a turning point z_0 of Q. Then $L(\alpha)$ must be also a boundary ray of T. In particular, $Q(z)\,dz^2$ is real negative on the interval $(0, z_0)$ of $L(\alpha)$ and real positive on (z_0, ∞), satisfying conditions (i) and (v) of Lemma 2.4.

Since T contains S and $\overline{T}$ does not contain anti-Stokes lines of P_m or P_d other than $L(\alpha)$, imaginary parts of $z^2 P_m$ and $z^2 P_d$ on the other boundary of T are both negative, satisfying condition (ii) of Lemma 2.4.

On $L(\alpha)$, $z^2 P_d$ is real positive and $z^2 P_m$ real negative. Let $L(\theta) \subset \overline{T} \setminus L(\alpha)$. The argument of P_d on $L(\theta)$ equals $(\theta - \alpha)(d + 2)$ and the argument of P_m on $L(\theta)$ equals $\pi + (\theta - \alpha)(m + 2)$. The difference of these two arguments cannot be $(2k + 1)\pi$, otherwise Q would have a turning point on $L(\theta)$. Also, neither of these two arguments can be $2k\pi$, otherwise $L(\theta)$ would be a ray of partition $\mathcal{T}(Q)$, and T would contain a sector of $\mathcal{T}(Q)$ other than S. Hence the argument of $z^2Q(z)$ on $L(\theta)$ is between these two arguments, and cannot be equal $2k\pi$, satisfying condition (iv) of Lemma 2.4.

The case when $L(\beta)$ is the Stokes line of P_d and contains a turning point of Q is treated similarly. This proves part (a) of Lemma 2.9.

If neither $L(\alpha)$ nor $L(\beta)$ contain turning points of Q, similar arguments prove part (b) of Lemma 2.9. □

Lemma 2.10. *Let $Q(z)$ be as in Definition 2.6. Each non-zero turning point of Q belongs either to a sector S of $\mathcal{T}(Q)$ satisfying*

(i) *S is bounded by a Stokes line and an anti-Stokes line of P_d,*
(ii) *$\mathrm{Im}(z^2P_d(z))$ and $\mathrm{Im}(z^2P_m(z))$ have opposite signs in S,*

or to the common boundary of two such sectors.

Conversely, every sector of $\mathcal{T}(Q)$ satisfying (i) *and* (ii) *has a non-zero turning point of $Q(z)\,dz^2$ either inside or on its boundary.*

Proof. The Stokes complex of a monomial P coincides with the set $\{z : z^2P(z) \in \mathbf{R}_+\}$. At a non-zero turning point p of Q, $P_d(p) = -P_m(p)$.

If $p^2P_d(p)$ and $p^2P_m(p)$ are real, p belongs either to a Stokes line of P_d which is also an anti-Stokes line of P_m, or to an anti-Stokes line of P_d which is also a Stokes line of P_m. Such a line is the boundary of two sectors of $\mathcal{T}(Q)$ satisfying conditions (i) and (ii) of Lemma 2.10. On any other boundary of a sector from $\mathcal{T}(Q)$, either $z^2P_d(z)$ and $z^2P_m(z)$ are both real of the same sign, or one of them is real and another is not, hence their sum $z^2Q(z)$ cannot vanish.

If $p^2P_d(p)$ and $p^2P_m(p)$ are not real, their imaginary parts should be opposite. Since $\mathrm{Im}(z^2P_d(z))$ and $\mathrm{Im}(z^2P_m(z))$ do not change sign inside any sector of $\mathcal{T}(Q)$, they have opposite signs in the sector S of $\mathcal{T}(Q)$ containing p. If $z^2P_d(z)$ and $z^2P_m(z)$ are both real on one of the two boundary rays of S, either they both map S to the upper half plane, or both map S to the lower half plane, or they map S to the opposite half planes. In the first two cases, their sum $z^2Q(z)$ cannot vanish in S. In the last case, the signs of $z^2P_d(z)$ and $z^2P_m(z)$ on a boundary ray of S should be opposite, hence there is a non-zero turning point on the boundary of S. Since S is a sector with the angle $\pi/(d+2)$, and the angle between turning points of Q is $2\pi/(d-m)$, there are no turning points inside S.

If the sector S of $\mathcal{T}(Q)$ containing a turning point does not have a boundary where both $z^2P_d(z)$ and $z^2P_m(z)$ are real, it may be bounded either by a Stokes line and an anti-Stokes line of P_d, or by a Stokes line of P_d and a Stokes line of P_m, or else by an anti-Stokes line of P_d and an anti-Stokes line of P_m. Otherwise, the imaginary parts of

$z^2P_d(z)$ and $z^2P_m(z)$ would have the same sign in S. In the last two cases, imaginary parts of $P_m/P_d = z^{d-m}$ have the same sign on the two boundaries of S, hence imaginary part of z^{d-m} does not change sign in S, hence S cannot contain a turning point.

This completes the proof of the first part of Lemma 2.10.

Conversely, if a sector S satisfying conditions (i) and (ii) of Lemma 2.10 does not have a turning point on its boundary, we may assume that $S = S(\alpha, \beta)$ where $\beta = \pi/(d+2) + \alpha$, $z^2P_d(z)$ is real negative on $L(\alpha)$ and real positive on $L(\beta)$, and $\text{Im}(z^2P_m(z)) > 0$ in S. (The opposite case is treated by changing the signs of P_d and P_m.) Then $\text{Im}(P_d/P_m)$ is positive on $L(\alpha)$ and negative on $L(\beta)$, hence $S(\alpha, \beta)$ contains a point p with $P_d(p)/P_m(p) = -1$. □

Lemma 2.11. *A sector $S = S(\alpha, \beta)$ satisfying condition* (i) *of Lemma 2.10 satisfies its condition* (ii) *if and only if it is contained in a sector $S(\alpha', \beta')$, with $\alpha' - \beta' = \pi/(m+2)$, bounded by a Stokes line and an anti-Stokes line of P_m, with the order opposite to the order of the Stokes and anti-Stokes lines of P_d bounding S: either $L(\alpha)$ is a Stokes line of P_d and $L(\alpha')$ is an anti-Stokes line of P_m, or $L(\alpha)$ is an anti-Stokes line of P_d and $L(\alpha')$ a Stokes line of P_m.*

Proof. Suppose that $L(\alpha)$ is a Stokes line of P_d and $L(\beta)$ is its anti-Stokes line. Then $\text{Im}z^2P_d(z) < 0$ in S. Condition (i) of Lemma 2.10 implies that $\text{Im}z^2P_m(z) < 0$ in S. In particular, there are no Stokes or anti-Stokes lines of P_m inside of S. Hence S belongs to a sector $S(\alpha', \beta')$, with $\alpha' - \beta' = \pi/(m+2)$, such that $L(\alpha')$ is an anti-Stokes line of P_m and $L(\beta')$ is its Stokes line. The opposite statement, and the case when $L(\alpha)$ is an anti-Stokes line of P_d, are proved by similar arguments. □

Lemma 2.12. *Let $Q(z)$ be as in Definition 2.6. Let S be a stable sector of the partition $\mathcal{S}(Q)$. Then a sector $T \neq S$ of $\mathcal{S}(Q)$ adjacent to a boundary ray L of S does not contain anti-Stokes lines of P_d, unless L is a Stokes line of Q.*

Proof. If L is a Stokes line of Q then the partition $\mathcal{S}(Q)$ is symmetric with respect to L, and T is a stable sector. In particular, T contains an anti-Stokes line of P_d by Lemma 2.7. Otherwise, if T contains an anti-Stokes line of P_d, condition (a) of Definition 2.6 implies that L is a Stokes line of P_d. Lemma 2.11 then implies that T contains a turning point of Q. □

Theorem 2.13. *Let $Q(z)$ be as in Definition 2.6. A sector S of the partition $\mathcal{S}(Q)$ that is either stable or marginally stable does not intersect the Stokes complex of Q.*

Proof. Let L be a boundary ray of S, and let T be the sector of $\mathcal{S}(Q)$ adjacent to S and having a common boundary ray L with S. If L is not a Stokes line of Q, Lemma 2.10 and conditions (a) and (b) of Definition 2.6 imply that T satisfies conditions of Lemma 2.9. In particular, T is a trap for Q.

In the case (b) of Lemma 2.9, Corollary 2.3 implies that T contains a Stokes line of Q with one end at the origin and another at infinity. In the case (a) of Lemma 2.9, there is a turning point z_0 on the boundary ray M of T other than L. In this case, T contains a Stokes line of Q with one end at z_0 and another at infinity, and M contains a Stokes line of Q with one end at z_0 and another at the origin.

In any case, there are no turning points of Q in S, and no Stokes lines of Q inside S with an end at the origin, hence each vertical line Γ of Q in S may either go to infinity or exit S through one of the boundary rays.

If L is a Stokes line of Q, Γ cannot cross L. Otherwise, if Γ crosses L it enters the sector T adjacent to S. It cannot cross the Stokes lines contained in T (or on the boundary of T), cannot enter the turning point at the boundary of T, and cannot return to S. The only remaining option for Γ is having an end at infinity in T. □

Example 2.14. In Fig. 3 and Fig. 4, the Stokes complexes of $Q(z) = z^4 + iz^3$ and $Q(z) = z^4 + e^{\pi i/4}z^3$ are shown, with the corresponding partitions $\mathcal{S}(Q)$. The stable sectors in Fig. 1 (see Example 2.8) are exactly those having empty intersection with the Stokes complex in Figs. 3 and 4.

In Fig. 5 and Fig. 6, the Stokes complexes of $Q(z) = z^4 + z^3$ and $Q(z) = z^3 - z^2$ are shown, with the corresponding partitions $\mathcal{S}(Q)$. The stable and marginally stable sectors in Fig. 2 (see Example 2.8) are exactly those having empty intersection with the Stokes complex in Figs. 5 and 6.

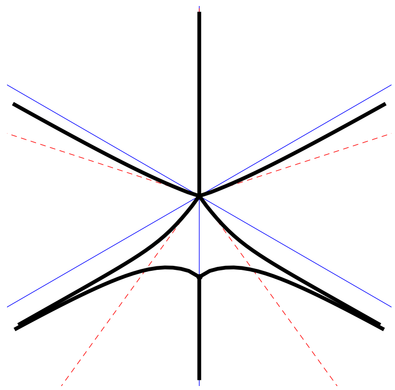

Fig. 3. Stokes complex of $z^4 + iz^3$.

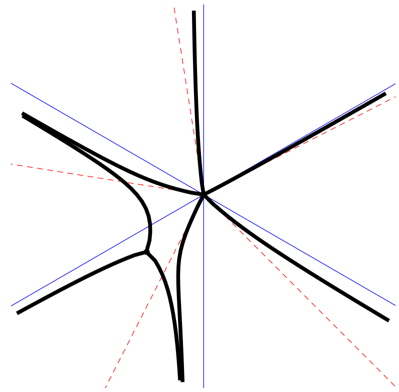

Fig. 4. Stokes complex of $z^4 + e^{\pi i/4} z^3$.

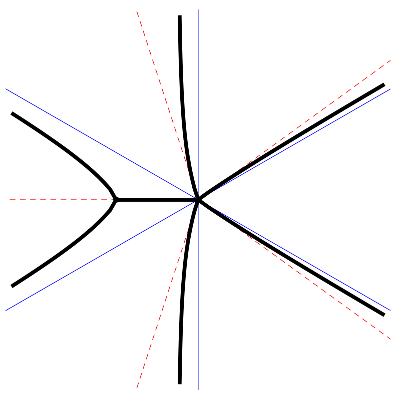

Fig. 5. Stokes complex of $z^4 + z^3$.

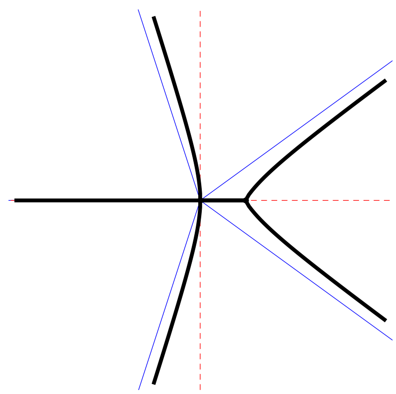

Fig. 6. Stokes complex of $z^3 - z^2$.

Remark 2.15. A binomial potential may have no stable sectors. Figs. 7 and 8 show the Stokes complexes of $Q(z) = \pm z^6 + z$.

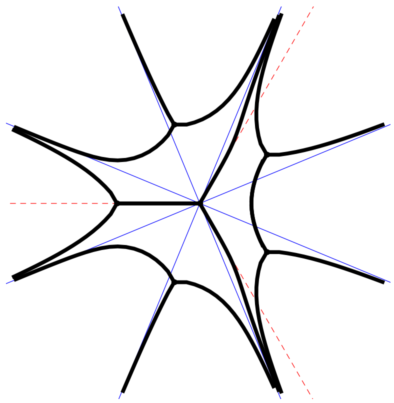

Fig. 7. Stokes complex of $z^6 + z$.

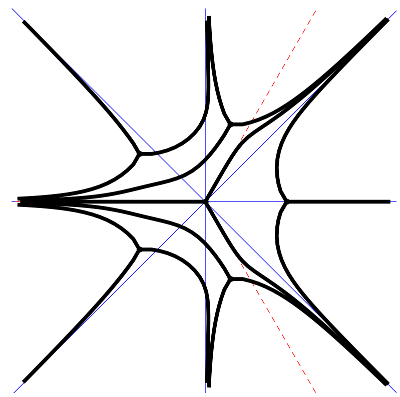

Fig. 8. Stokes complex of $-z^6 + z$.

Lemma 2.16. *Let $Q(z) = z^d + cz^m$ with $c \neq 0$, and let $q(z) = a_m z^m + \cdots + a_0$. Let $S = S(\alpha, \beta)$ be any sector such that $\bar{S}$ does not contain non-zero turning points of Q. There exists $C = C(c, \alpha, \beta) > 0$ such that, for every $\varepsilon > 0$ and $r > 0$, and for all $a_0, \ldots, a_m$ with $|a_j| < \varepsilon r^{m-j}$, we have $|q(z)/Q(z)| < \varepsilon C/(1 + |z|^{d-m})$ for all $z \in S_r(\alpha, \beta)$.*

Proof. We have $|Q(z)| > C_1(|z|^m + |z|^d)$ for $z \in S$. Conditions on the coefficients a_j imply that $|q(z)| < \varepsilon(m+1)|z|^m$ in $S_r(\alpha, \beta)$. Hence $|q(z)/Q(z)| < \varepsilon C/(1 + |z|^{d-m})$ in $S_r(\alpha, \beta)$, where $C = (m+1)C_1$. □

Corollary 2.17. *Let $Q(z) = tz^d + cz^m$ with $t > 0$ real and $c \neq 0$. Let $q(z) = a_m z^m + \cdots + a_0$. Let $S = S(\alpha, \beta)$ be any sector such that $\bar{S}$ does not contain non-zero turning points of Q. There exists $C = C(c, \alpha, \beta) > 0$ such that, for every $\varepsilon > 0$ and $R > 0$, and for all $a_0, \ldots, a_m$ with $|a_j| < \varepsilon R^{m-j}$, we have $|q(z)/Q(z)| < \varepsilon C/(1 + t|z|^{d-m})$ for all $z \in S_R(\alpha, \beta)$.*

Proof. Let $z = ut^{1/(m-d)}$. In the new variable u, we have $\hat{Q}(u) = Q(z)t^{m/(d-m)} = u^d + cu^m$, $\hat{q}(u) = q(z)t^{m/(d-m)} = \sum_{j=0}^{m} a_j t^{(m-j)/(d-m)} u^j$. We can apply Lemma 2.16 to $\hat{Q}(u)$, $\hat{q}(u)$ and $r = Rt^{1/(d-m)}$. □

Lemma 2.18. *Let $Q(z) = tz^d + cz^m$ with $t > 0$ real and $c \neq 0$. Let $q(z) = a_m z^m + \cdots + a_0$. Let S be either a sector $S(\alpha, \beta)$ of the partition $\mathcal{S}(Q)$ such that $\overline{S}$ does not contain anti-Stokes lines of z^d or, for $\beta = \alpha$, a common Stokes line $L(\alpha) = L(\beta)$ of z^d and cz^m. Let T be any sector containing $\overline{S} \setminus \{0\}$ such that $\overline{T}$ does not contain anti-Stokes lines of z^d and cz^m. Then there exist $\varepsilon = \varepsilon(c, T) > 0$ and $\eta = \eta(c, T) > 0$ such that, for every $R > 0$, and for all $a_0, \ldots, a_m$ with $|a_j| < \varepsilon R^{m-j}$, T_R is an η-trap for $Q + q$.*

Proof. Lemma 2.10 implies that T does not contain turning points of Q. The statement now follows from part (b) of Lemma 2.9, since the vertical directions of $Q+q$ are ε-close to those of Q in T_R. □

Corollary 2.19. *Let Q and q be as in Lemma 2.18. Let $S = S(\alpha, \beta)$ be a stable sector of Q, and let $T = S(\alpha + \delta, \beta - \delta)$, for $\delta > 0$, be a subsector of S containing both anti-Stokes lines of the monomials of Q. Then there is $\varepsilon = \varepsilon(c, T) > 0$ and $\eta = \eta(c, T) > 0$ such that, for every $R > 0$ and for all $a_0, \ldots, a_m$ with $|a_j| < \varepsilon R^{m-j}$, T_R is an η-trap for $-(Q + q)$.*

Proof. From the definition of a stable sector, T satisfies conditions of Lemma 2.18 for $-Q$. □

Example 2.20. A stable sector S for $Q = z^4 + iz^3$ (see Fig. 1a) and its subsector T satisfying conditions of Corollary 2.19, are shown in Fig. 9.

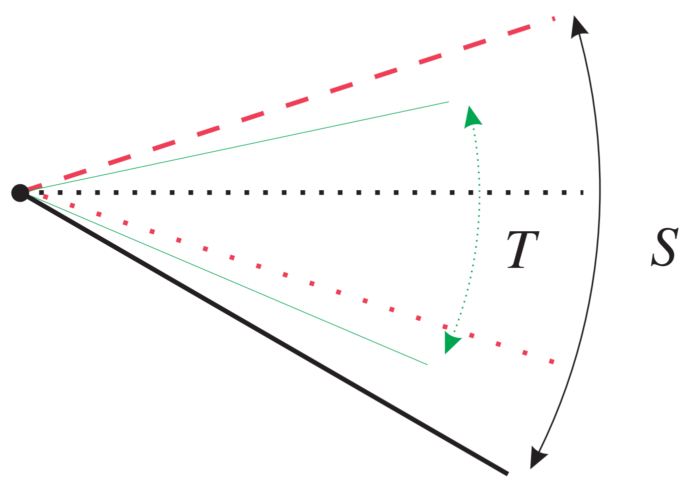

Fig. 9. Subsector T satisfying Corollary 2.19 of a stable sector S for $Q = z^4 + iz^3$

Theorem 2.21. *Let $Q(z) = z^d + cz^m$ with $c \neq 0$, and let $q(z) = a_m z^m + \cdots + a_0$. Let $S = S(\alpha, \beta)$ be a stable sector of Q. For every $\delta > 0$, there exists $\varepsilon = \varepsilon(c, \delta) > 0$ such that, for every $r > 0$, the set $S_r(\alpha + \delta, \beta - \delta)$ has empty intersection with the Stokes complex of $Q + q$ for all $a_0, \ldots, a_m$ with $|a_j| < \varepsilon r^{m-j}$.*

Proof. Let $S(\alpha', \alpha)$ and $S(\beta, \beta')$ be the two sectors of the partition $\mathcal{S}(Q)$ adjacent to S. Define $T = S(\alpha' - \delta, \alpha + \delta)$, $T' = S(\beta - \delta, \beta' + \delta)$, unless either $L(\alpha)$ or $L(\beta)$ is a Stokes line of Q. If $L(\alpha)$ (resp. $L(\beta)$) is a Stokes line of Q, define $T = S(\alpha - \delta, \alpha + \delta)$ (resp. $T' = S(\beta - \delta, \beta + \delta)$).

Since S is a stable sector, $\overline{S(\alpha',\alpha)}$ (resp. $\overline{S(\beta,\beta')}$) does not contain anti-Stokes lines of z^d and cz^m, unless $L(\alpha)$ (resp. $L(\beta)$) is a Stokes line of Q. We can assume that δ is small enough, so that the sectors $\overline{T}$ and $\overline{T'}$ do not contain anti-Stokes lines of z^d and cz^m. Lemma 2.10 implies that $\overline{T}$ and $\overline{T'}$ do not contain non-zero turning points of Q. Lemma 2.18 implies that there exist $\varepsilon > 0$ and $\eta > 0$ such that T_r and T_r' are η-traps for $Q+q$, for all $r > 0$ and all $a_0, \ldots, a_k$ with $|a_j| < \varepsilon r^{m-j}$.

Since the intersection angles between the vertical directions of Q in $S(\alpha+\delta, \beta-\delta)$ and the rays through the origin are separated from 0, every vertical line Γ of Q in $S(\alpha+\delta,\beta-\delta)$ has both its ends on the rays $L(\alpha+\delta)$ and $L(\beta-\delta)$. Moreover, there is a constant $C_1 > 0$ such that the distance ρ from Γ to the origin and the distance σ from its intersections with any of the rays $L(\alpha+\delta)$ and $L(\beta-\delta)$ to the origin satisfy $C_1\sigma < \rho < \sigma$.

Since $|q(z)/Q(z)| < \varepsilon C/(1+|z|^{d-m})$ in $S_r(\alpha+\delta,\beta-\delta)$ when $|a_j| < \varepsilon r^{m-j}$, this implies that, for small positive ε and every vertical line Γ' of $Q+q$ having non-empty intersection with $S_r(\alpha+\delta,\beta-\delta)$, the corresponding distances ρ' and σ' satisfy $C_2 r < \rho' < \sigma'$ for some $C_2 > 0$. In particular, Γ' cannot enter any turning points of $Q+q$ close to 0, since they have absolute values of the order εr.

Since T_r and T_r' are η-traps for $Q+q$, extensions of Γ' to the sectors T and T', across $L(\alpha+\delta)$ and $L(\beta-\delta)$, remain in these sectors and have the ends at infinity. In particular, they cannot enter any turning points of $Q+q$, hence Γ' does not belong to the Stokes complex of $Q+q$. □

Corollary 2.22. *Let $Q(z) = tz^d + cz^m$ with $t > 0$ real and $c \neq 0$. Let $S = S(\alpha,\beta)$ be a stable sector of Q. Let $q(z) = a_m z^m + \cdots + a_0$. For each $\delta > 0$, there exists $\varepsilon = \varepsilon(c,\delta)$ such that, for each $R > 0$, $S_R(\alpha+\delta,\beta-\delta)$ has empty intersection with the Stokes complex of $Q+q$ for all $a_0,\ldots,a_m$ with $|a_j| < \varepsilon R^{m-j}$.*

Proof. Let $z = ut^{1/(m-d)}$. In the new variable u, we have $\hat{Q}(u) = Q(z)t^{m/(d-m)} = u^d + cu^m$, $\hat{q}(u) = q(z)t^{m/(d-m)} = \sum_{j=0}^{d} a_j t^{(m-j)/(d-m)}$. We can apply Theorem 2.21 to $\hat{Q}(u)$, $\hat{q}(u)$ and $r = Rt^{1/(d-m)}$. □

Case $m = d-1$. Rescaling Q and z, we can assume $Q(z) = z^d + e^{i\theta} z^{d-1}$. Adding $2\pi/(d+2)$ to θ is equivalent to rotating the Stokes complex of Q, and the partitions $\mathcal{S}(Q)$ and $\mathcal{T}(Q)$, by $2\pi/(d+2)$ counterclockwise. Changing the sign of θ is equivalent to reflecting the Stokes complex of Q about the real axis. Hence it is enough to consider the case $0 \leq \theta \leq \pi/(d+2)$.

Lemma 2.23. *Let* $Q(z) = z^d + e^{i\theta} z^{d-1}$.

(i) *For* $0 < \theta < \pi/(d+2)$, *and for* $\theta = 0$ *when* d *is odd, conditions of Lemma 2.9 (and Lemma 2.2 with* $R = 0$*) are satisfied for the following* $d+1$ *sectors of the partition* $\mathcal{S}(Q)$*:*

$$S((2k+1)\pi/(d+2),\ ((2k+1)\pi-\theta)/(d+1)) \quad \text{for } k = 0,\dots,\lfloor d/2\rfloor, \tag{2.1}$$

$$S(((2k+1)\pi-\theta)/(d+1),\ (2k+1)\pi/(d+2)) \quad \text{for } k = -1,\dots,-\lfloor (d+1)/2\rfloor. \tag{2.2}$$

(ii) *For* $\theta = \pi/(d+2)$, *the sector with* $k = 0$ *in (2.2) degenerates into a Stokes line* $L(\pi/(d+2))$ *of* Q.

(iii) *For even* d *and* $\theta = 0$, *conditions of Lemma 2.9 are satisfied for all sectors in (2.2) and (2.3) except the sector with* $k = d/2$ *in (2.2).*

(iv) *For odd* d *and* $\theta = \pi/(d+2)$, *conditions of Lemma 2.9 are satisfied for all non-degenerate sectors in (2.2) and (2.3) except the sector with* $k = -(d+1)/2$ *in (2.3).*

Example 2.24. The Stokes complexes in Fig. 4 (for $d = 4$) and Figs. 10 and 11 (for $d = 3$) illustrate the statement (i) of Lemma 2.23. The Stokes complex in Fig. 4 corresponds to $\theta = \pi/4 = \pi/12 - \pi/3$. It can be obtained from the Stokes complex for $\theta = \pi/12$ by the $\pi/3$ clockwise rotation. The Stokes complex in Fig. 10 corresponds to $\theta = 0$. The Stokes complex in Fig. 11 corresponds to $\theta = \pi/2 = \pi/10 + 2\pi/5$. It can be obtained from the Stokes complex for $\theta = \pi/10$ by the $2\pi/5$ counterclockwise rotation.

The Stokes complexes in Fig. 3 (for $d = 4$) and Fig. 6 (for $d = 3$) illustrate the statement (ii) of Lemma 2.23. The Stokes complex in Fig. 3 corresponds to $\theta = \pi/2 = \pi/6 + \pi/3$. It can be obtained from the Stokes complex for $\theta = \pi/6$ by the $\pi/3$ counterclockwise rotation. The Stokes complex in Fig. 6 corresponds to $\theta = \pi = \pi/5 + 4\pi/5$. It can be obtained from the Stokes complex for $\theta = \pi/5$ by the $4\pi/5$ counterclockwise rotation.

The Stokes complex in Fig. 5 illustrates the statement (iii) of Lemma 2.23 for $d = 4$.

The Stokes complex in Fig. 6 illustrates the statement (iv) of Lemma 2.23 for $d = 3$. It corresponds to $\theta = \pi$, and can be obtained from the Stokes complex for $\theta = \pi/5$ by the $4\pi/5$ counterclockwise rotation.

Figs. 5 and 6 illustrate the statement of Lemma 2.25 below.

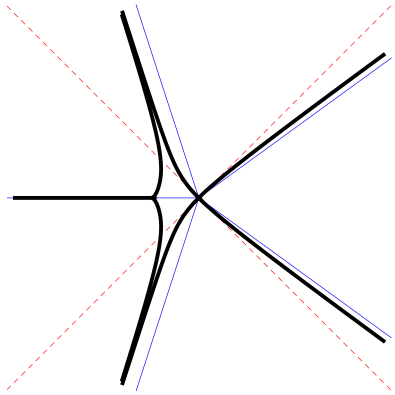

Fig. 10. Stokes complex of $z^3 + z^2$.

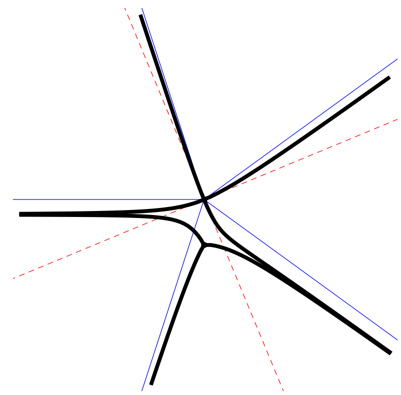

Fig. 11. Stokes complex of $z^3 + iz^2$.

Lemma 2.25. *For even d and $\theta = 0$ and for odd d and $\theta = \pi/(d+2)$, the Stokes complex of Q contains a segment connecting 0 with the non-zero turning point. The other two Stokes lines with the ends at the non-zero turning point remain inside the sector*

$$S((d+1)\pi/(d+2),\ (d+3)\pi/(d+2)) \text{ for even } d, \tag{2.3}$$

$$S(\pi, (d+4)\pi/(d+2)) \text{ for odd } d. \tag{2.4}$$

Lemma 2.26. *For $0 < \theta < \pi/(d+2)$, and for $\theta = 0$ when d is odd, the following d sectors are stable:*

$$\begin{aligned} & S(-\pi/(d+2),\ \pi/(d+2)), \\ & S(((2k-1)\pi - \theta)/(d+1),\ (2k+1)\pi/(d+2)) \\ & \text{for } k = 1, \ldots, \lfloor d/2 \rfloor, \end{aligned} \tag{2.5}$$

$$\begin{aligned} & S((2k-1)\pi/(d+2),\ ((2k+1)\pi - \theta)/(d+1)) \\ & \text{for } k = -1, \ldots, -\lfloor (d-1)/2 \rfloor. \end{aligned} \tag{2.6}$$

Lemma 2.27. *For even d and $\theta = 0$, the sectors in (2.5,2.6,2.7) are stable, except the sector*

$$S((d-1)\pi/(d+1),\ (d+1)\pi/(d+2)) \tag{2.7}$$

corresponding to $k = d/2$ in (2.7). The sectors (2.7) and

$$S(-(d+1)\pi/(d+2),\ -(d-1)\pi/(d+1)) \tag{2.8}$$

are marginally stable.

For odd d and $\theta = \pi/(d+2)$, the sectors in (2.5,2.6,2.7) are stable, except the sector

$$S(-d\pi/(d+2),\ -((d-2)\pi + \pi/(d+2))/(d+1)) \tag{2.9}$$

corresponding to $k - \lfloor (d-1)/2 \rfloor$ *in (2.7). The sectors (2.9) and*

$$S((d-1)\pi/(d+1),\, \pi) \tag{2.10}$$

are marginally stable.

Case $m = d - 2$**.** Let $Q(z) = z^d + cz^{d-2}$. Examples of the Stokes complexes for $d = 4$ and $d = 3$ are shown in Figs. 12-17. Generically (as in Figs. 12, 14, 17) there are $d - 2$ stable sectors. For d even, if there are Stokes lines of Q connecting the origin with the turning points, there are $d - 4$ stable and 4 marginally stable sectors. For d odd, if there is a Stokes line of Q connecting the origin with a turning point, there are $d - 3$ stable and 2 marginally stable sectors.

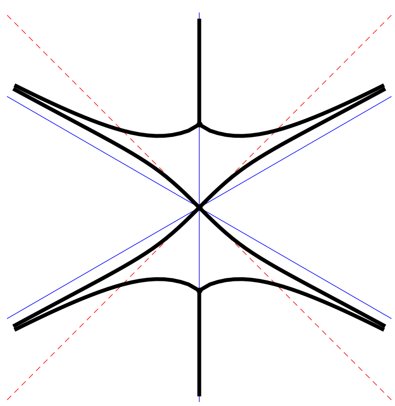

Fig. 12. Stokes complex of $z^4 + z^2$.

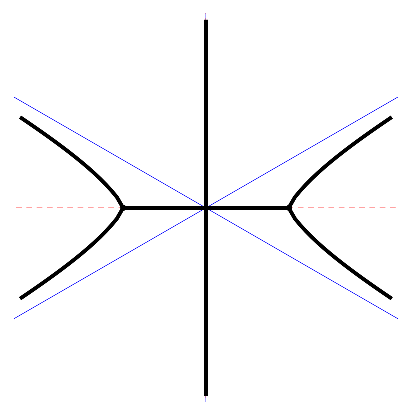

Fig. 13. Stokes complex of $z^4 - z^2$.

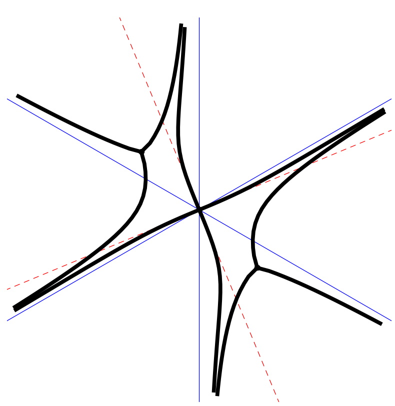

Fig. 14. Stokes complex of $z^4 + iz^2$.

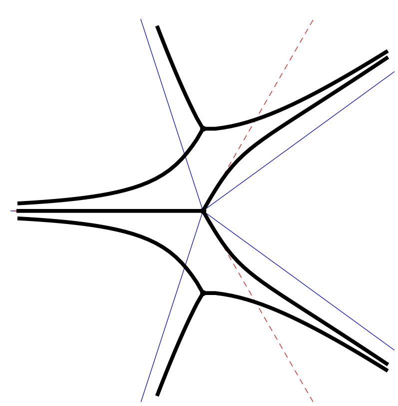

Fig. 15. Stokes complex of $z^3 + z$.

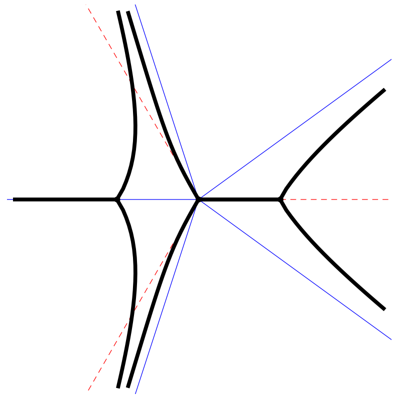

Fig. 16. Stokes complex of $z^3 - z$.

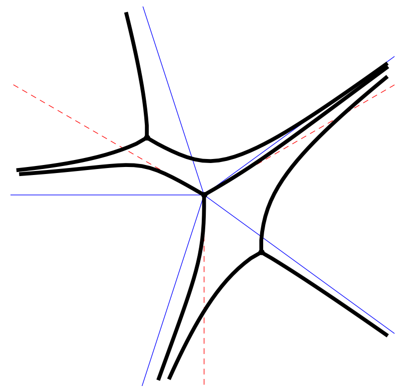

Fig. 17. Stokes complex of $z^3 + iz$.

3. Perturbation of eigenvalues and eigenfunctions

We consider differential equations

$$-y'' + Q_t(z)y = 0, \tag{3.1}$$

where $Q_t(z) = tz^d + a_{d-1}z^{d-1} + \ldots + a_q z - \lambda$ is a polynomial of degree d. The *separation rays* are defined by

$$\mathrm{Re}\left(\int_0^z \sqrt{t\zeta^d}d\zeta\right) = 0, \quad \text{that is} \quad tz^{d+2} < 0.$$

These divide the plane into $d + 2$ open sectors S_j which we call *Stokes sectors*. We enumerate these sectors by residues modulo $d + 2$ counterclockwise, starting from an arbitrary sector S_0. We use the notation $S_j = \{re^{i\theta} : \theta \in A_j\}$, where $A_j = (\theta_j^-, \theta_j^+)$ is an open arc of the unit circle corresponding to S_j.

We say that a ray has a *separation direction* if it is parallel to one of the separation rays.

A solution y of (3.1) is called *subdominant* in S_j if $y(re^{i\theta}) \to 0$ as $r \to \infty$ for $\theta \in A_j$. If this condition holds for one $\theta \in A_j$ then it also holds for all $\theta \in A_j$. For every sector S_j, the subspace of solutions which are subdominant in S_j is one-dimensional.

Let S_j and S_k be two *non-adjacent* sectors, that is $k \neq j \pm 1 \bmod (d+2)$. We impose the boundary conditions:

$$y \text{ is subdominant in } S_j \text{ and in } S_k. \tag{3.2}$$

The boundary value problem (3.1), (3.2) has discrete spectrum with an infinite sequence of eigenvalues tending to infinity. All eigenspaces are one-dimensional.

The eigenvalues λ can be found from the equation

$$F_t(\mathbf{a}, \lambda) = 0,$$

where F is the so-called spectral determinant, which is an entire function of $\mathbf{a} = (a_{d-1}, \dots, a_1)$ and λ when $t \neq 0$ is fixed. The dependence of F on t is analytic for $t \neq 0$ but $t = 0$ is a singularity of F.

Here we study this singularity. Much is known on this question for the boundary value problems

$$-w'' + (\epsilon z^d + z^2)w = \lambda w \tag{3.3}$$

with normalization conditions on the real axis, [8, 39, 11, 26]. Our method is different from the methods of these authors: we use the theory of differential equations in the complex domain rather than the general perturbation theory of linear operators. We obtain a simple sufficient condition of continuity of spectrum which is applicable to a general class of boundary value problems (3.1), (3.2). However, for the case (3.3) our conditions of continuity of spectra are more restrictive than those given in [39, 11].

We consider a one-parametric family

$$-y'' + (P_t(z, \mathbf{a}) - \lambda)y = 0 \tag{3.4}$$

where $\mathbf{a} = (a_1, \dots, a_{m-1})$, and

$$P_t(z, \mathbf{a}) = tz^d + c(t)z^m + a_{m-1}z^{m-1} + \dots + a_1 z, \tag{3.5}$$

where $d > m$ are positive integers, $t \geq 0$, and $c(t)$ is a continuous function, $c(0) \neq 0$.

We say that two rays L_1 and L_2 are *admissible* for (3.4) if for every $t \in [0, t_0]$ they have non-separation directions, and belong to two non-adjacent Stokes sectors. Then the boundary conditions (3.2) can be stated as

$$y(z) \to 0 \quad \text{as} \quad z \to \infty, \quad z \in L_1 \cup L_2. \tag{3.6}$$

Asymptotic behavior of solutions of the differential equation (3.1) as $z \to \infty$ depends on the properties of the quadratic differential $Q_t(z)dz^2$. We use the terminology introduced in the beginning of Section 2. The Stokes complex is the 1-skeleton of the cell decomposition of the Riemann sphere whose vertices are turning points and ∞, and edges are the Stokes lines. The 2-cells of this decomposition are called the *faces* or the *Stokes regions.* Stokes lines which have the point ∞ in their closures are asymptotic to the separation rays as $z \to \infty$. There can be several Stokes lines asymptotic to the same ray.

Stokes regions can be of two types: *half-plane type* and *strip type* regions. Each half-plane type region is bounded by a single simple curve whose ends are at infinity and have two adjacent Stokes directions.

Each Stokes region of strip type is bounded by two simple disjoint curves $\gamma_1(s)$ and $\gamma_2(s)$, $s \in (-\infty, \infty)$ such that, for $s \to +\infty$, γ_1 and γ_2 have the same separation direction θ', and for $s \to -\infty$ they have the same separation direction $\theta'' \neq \theta'$.

For all these properties of the Stokes complexes we refer to [23].

We denote by $\mathrm{St}(t, \mathbf{a}, \lambda)$ the Stokes complex of the equation (3.1) with $Q_t(z) = P_t(z, \mathbf{a}) - \lambda$, where P_t is given by (3.5).

Definition 3.1. We say that the spectrum of the problem (3.1), (3.4) is *continuous* at $t = 0$ if the following holds:

For every compact set $K = K_\lambda \times K_\mathbf{a} \subset \mathbf{C}_\lambda \times \mathbf{C}_\mathbf{a}^{m-1} = \mathbf{C}^m$, and every eigenvalue $\lambda_0 \in K_\lambda$ of (3.4), (3.6) with $t = 0$, and for every $\varepsilon \in (0, \varepsilon_0)$, there exists $\delta > 0$ such that for $t \in [0, \delta]$ problem (3.4), (3.6) has a unique eigenvalue λ_t satisfying

$$|\lambda_0 - \lambda_t| < \varepsilon, \tag{3.7}$$

and the convergence $\lambda_t \to \lambda_0$ is uniform with respect to $\mathbf{a} \in K_\mathbf{a}$. Here ε_0 is one half of the distance from λ_0 to the rest of the spectrum for $t = 0$.

Moreover, for every $t \in [0, \delta]$ and every eigenvalue $\lambda_t \in K_\lambda$ of (3.4), (3.6) there exists an eigenvalue λ_0 of (3.4), (3.6) with $t = 0$ such that (3.7) is satisfied.

Our criterion of continuity of the spectrum uses the notion of a stable sector, see Definition 2.6.

Theorem 3.2. *Suppose that the rays L_1 and L_2 are admissible for all $t \geq 0$, and belong to distinct stable sectors of the binomial $tz^d + c(0)z^m$. Then the spectrum of (3.1), (3.4) is continuous at $t = 0$.*

To prove this theorem, it is sufficient to show that the spectral determinant $F_t(\mathbf{a}, \lambda)$ converges to $F_0(\mathbf{a}, \lambda)$ as $t \to 0$ uniformly on compact subsets of the $(\mathbf{a}, \lambda)$-space. To study this spectral determinant we use the method of Sibuya [38]. Consider the equation (3.1), and let $L = (z_1, \infty)$ be a ray whose direction is not a separation direction, and which does not intersect the Stokes complex. It follows that L is completely contained in a Stokes region D_L of half-plane type.

Crucial role will be played by the multivalued function

$$\phi(z) = \int_{z_0}^{z} \sqrt{Q_t(\zeta)} d\zeta. \tag{3.8}$$

The singularities (algebraic branch points) of this function are at the turning points.

Every Stokes region is simply connected, so the branches of (3.8) are single-valued in D_L. There is a unique branch ϕ_L in D_L normalized by the conditions

$$\phi_L(z_0) = 0, \quad \operatorname{Re}\phi_L(z) \to +\infty, \quad z \in L, \quad z \to \infty,$$

where z_0 is a point in the closure of D_L.

Every branch of ϕ in every Stokes region of half-plane type maps this region conformally onto some right or left half-plane. Our branch ϕ_L maps D_L onto a right half-plane. If $z_0 \in \partial D_L$ then the image of D_L under ϕ_L is the right half-plane $\{\zeta : \operatorname{Re}\zeta > 0\}$.

Let us also choose a branch $(Q_t^{1/4})_L$ of $Q_t^{1/4}$ in D_L.

A theorem of Sibuya [38, Thm. 6.1] says that there exists a unique solution y_L of (3.1) normalized by the following asymptotics

$$y_L(z) = (1 + o(1))(Q_t^{-1/4})_L(z) \exp(-\phi_L(z)), \quad z \to \infty, \quad z \in L. \tag{3.9}$$

We call y_L the *Sibuya solution* associated with the ray L. Notice that Sibuya solution does not change when the ray L varies continuously if in the process it never intersects the Stokes complex, and never has a separation direction. In particular, y_L does not change when we shorten the ray L by removing a bounded initial segment of it.

Sibuya solution depends analytically on all coefficients of Q_t except the top one.

Theorem 3.3. *Let t_0 be a positive number, and let $K \subset \mathbf{C}^m$ be a compact set. Suppose that for $t \in [0, t_0]$ the ray L does not have a separation direction and belongs to a stable sector of $tz^d + c(0)z^m$. Then the Sibuya solution y_L of (3.4) depends continuously on $t \in [0, t_0]$. This means that for $t \to 0$, and every compact $K' \subset \mathbf{C}$ we have $y_L(z, t, \mathbf{a}, \lambda) \to y_L(z, 0, \mathbf{a}, \lambda)$ uniformly with respect to $(z, \mathbf{a}, \lambda) \in K' \times K$.*

The case $d = m + 1$ and $c \equiv c(0) > 0$ follows from a result of Sibuya [38, Thm. 12.1].

Theorem 3.2 is an easy consequence of Theorem 3.3. Indeed, λ is an eigenvalue of the problem (3.4), (3.6) if and only if $y_{L_1'}$ is proportional to $y_{L_2'}$ for some rays L_1' and L_2' which are parts of L_1 and L_2, respectively. This is equivalent to

$$W(y_{L_1'}, y_{L_2'})(0) = 0,$$

where W is the Wronskian determinant (with respect to the independent variable z) of the two solutions. It follows from Theorem 3.3 that this Wronskian is an entire function of $(\mathbf{a}, \lambda)$ which varies continuously with t (in the topology of uniform convergence on compact subsets), and this implies that the spectrum depends continuously on t. Here we used the fact that for every $(t, \mathbf{a}, c)$, $\partial W(0)/\partial\lambda \not\equiv 0$, see [38, Ch. 6].

Proof. In our construction of the Sibuya solution y_L we follow [30]. Let $D_L = D_L(t, \mathbf{a}, \lambda)$ be the Stokes region that contains L. Notice that L does not depend on $(t, \mathbf{a}, \lambda)$ while D_L does.

We make the change of the independent variable $\zeta = \phi_L(z), z = \psi_L(\zeta)$, ϕ_L is defined in (3.8) where we set $Q_t = P_t - \lambda$, and $\psi_L = \phi_L^{-1}$ is a conformal map of the right half-plane H onto D.

We explain the choice of the point z_0. Consider the horizontal trajectory from z_1, the beginning point of our ray L, towards the boundary of D_L. This trajectory intersects ∂D_L at some point v. Let γ be the segment of this trajectory from v to z_1. Consider the quantity

$$\phi_L(z_1) - \phi_L(v) = \int_\gamma \sqrt{Q_t(\zeta)}d\zeta > 0.$$

which is independent of the choice of z_0 in (3.8). It is clear that this integral is a continuous function of $(t, \mathbf{a}, \lambda)$ so there exists $\delta > 0$ such that

$$\phi_L(z_1) - \phi_L(v) > 2\delta$$

for all $(t, \mathbf{a}, \lambda) \in [0, t_0] \times K$. Let $z_0 \in \gamma$ be the unique point such that

$$\phi_L(z_1) - \phi_L(z_0) = \delta.$$

We choose this z_0 as a normalization point in the definition of ϕ_L (3.8).

With such choice of z_0, $\phi_L(z_0) = 0$, and ϕ_L maps a part of D_L onto the right half-plane $H = \{\zeta : \operatorname{Re}\zeta > 0\}$ and we have

$$\phi_L(z_1) = \delta > 0 \tag{3.10}$$

for all $(t, \mathbf{a}, \lambda) \in [0, t_0] \times K$.

Then $w(\zeta) = y \circ \psi(\zeta)$ satisfies

$$w'' + \left(\frac{\phi''}{\phi'^2} \circ \psi\right) w' = w,$$

see [30]. To kill the w' term we then set $w = bu$ where

$$b(\zeta) = (Q_t^{-1/4})_L \circ \psi(\zeta),$$

with any fixed branch $(Q_t^{1/4})_L$ of $Q_t^{1/4}$ which is specified by choosing one of the four possible values at the point z_0 continuously with respect to $(t, \mathbf{a}, \lambda) \in [0, t_0] \times K$. Then we obtain

$$u'' = (1 - g(\zeta))u, \tag{3.11}$$

where

$$g(\zeta)=\left(\frac{5}{16}\frac{Q_t'^2}{Q_t^3}-\frac{1}{4}\frac{Q_t''}{Q_t^2}\right)\circ\psi(\zeta). \tag{3.12}$$

The equation (3.11) is equivalent to the integral equation

$$u(\zeta)=e^{-\zeta}+\int_\zeta^\infty\left(e^{\zeta-\eta}-e^{-\zeta+\eta}\right)g(\eta)u(\eta)d\eta, \tag{3.13}$$

where the path of integration is a curve in the right half-plane on which $\mathrm{Re}\,\zeta\to+\infty$. We choose this curve in a special convenient way.

According to Corollary 2.19, there is an $R>0$ which depends only on c and K (but not on t), and a horizontal trajectory J in in $D_L\cap\{z:|z|<R\}$ which intersects all rays from the origin under an angle less than $\pi/2-\eta$, $\eta>0$, and

$$|\arg z-\arg z_k|\geq\delta,\quad z\in J, \tag{3.14}$$

for all turning points z_k such that $|z_k|>R$. Our map ϕ_L sends this trajectory to a horizontal ray which we call ℓ. The path of integration in (3.13) goes from ζ to some ζ_0 on the ray ℓ, and then follows the ray ℓ to infinity in the right half-plane.

This integral equation is solved by successive approximation. We set $u(\zeta)=v(\zeta)e^{\zeta}$, and obtain

$$v(\zeta)=1+\frac{1}{2}\int_\zeta^\infty\left(e^{2(\zeta-\eta)}-1\right)g(\eta)v(\eta)d\eta, \tag{3.15}$$

which we abbreviate as $v=1+F(v)$. Setting $v_0=0$ and $v_{n+1}=1+F(v_n)$, we obtain

$$\|v_{n+1}-v_n\|_\infty\leq\|v_n-v_{n-1}\|\int_\zeta^\infty|g(t)|dt.$$

Now, if $\zeta>0$ is large enough, we have

$$\int_\zeta^\infty|g(\eta)|d\eta<1/2 \tag{3.16}$$

for all values of parameters $t\in[0,t_0],(a,\lambda)\in K$. We state this as a lemma:

Lemma. *There exists b such that for every $t\in[0,t_0]$ and $(a,\lambda)\in K$ we have*

$$\left|\int_{J_b}\left(\frac{5}{16}\left(\frac{Q_t'}{Q_t}\right)^2-\frac{1}{4}\frac{Q_t''}{Q_t}\right)Q_t^{-1/2}dz\right|<1/2.$$

Here J_b is a part of the curve J from $b\in J$ to infinity.

The integral in this lemma equals to the integral in (3.16) by the change of the variable $z = \psi(\zeta)$, $\sqrt{Q_t(z)}dz = d\zeta$.

Proof. Let $z_1, \ldots, z_d$ be all zeros of Q_t listed with multiplicity, in an order of non-decreasing moduli. Suppose that $z_1, \ldots, z_M$ are in the disc $|z| < R$ while the rest are outside. We have

$$\frac{Q_t'}{Q_t} = \sum_{k=1}^{M} \frac{1}{z - z_k} + \sum_{k=M+1}^{d} \frac{1}{z - z_k} = \sigma_1(z) + \sigma_2(z),$$

and

$$\frac{Q_t''}{Q_t} = \left(\frac{Q_t'}{Q_t}\right)' + \left(\frac{Q_t'}{Q_t}\right)^2 = \sigma_1'(z) + \sigma_2'(z) + \sigma_1^2(z) + 2\sigma_1(z)\sigma_2(z) + \sigma_2^2(z).$$

First we estimate

$$|\sigma_1| \leq \frac{M}{|z| - R}, \quad |\sigma_1'(z)| \leq \frac{M}{(|z| - R)^2}. \tag{3.17}$$

To estimate σ_2 we first use (3.14) to conclude that

$$|z - z_k| \geq C_0|z|, \quad z \in J, \quad k \geq M \tag{3.18}$$

where C_0 depends only on δ. Then

$$|\sigma_2(z)| \leq C_1/|z|, \quad |\sigma_2'| \leq C_2/|z|^2. \tag{3.19}$$

Applying these inequalities, we obtain that

$$\left|\frac{5}{16}\frac{Q_t'^2}{Q_t^2} - \frac{1}{4}\frac{Q_t''}{Q_t}\right| \leq \frac{C}{|z|^2},$$

on J, where C depends only on δ.

Now we write

$$|Q_t(z)| = t\prod_{j=1}^{M}|z - z_j| \prod_{j=M+1}^{d} |z - z_j| \geq tC_3^d(|z| - R|)^M \prod_{j=M+1}^{d} |z_j|.$$

Here we used inequality (3.18) with interchanged z and z_k. It is easy to see by Vieta's theorem that

$$\prod_{j=M+1}^{d} |z_j| \geq c_4 t^{-1},$$

where c_4 depends only on K and $c(0)$ This shows that $|Q_t(z)| \leq C_6|z|^M$.

Now we use the fact that the angle between J and the radial direction is less than $\pi/2 - \eta$. This gives $|dz| \leq C_7 d|z|$ with C_7 depending only on δ.

Putting all this together we conclude that our integral is majorized by the integral

$$\int_{|b|}^{\infty} |z|^{-2-M/2} d|z|$$

which proves the lemma. □

So the series $\sum v_n$ is convergent uniformly in $\zeta > A$ for some $A > 0$ and this convergence is uniform with respect to t, a, λ. Then an application of the theorem on uniqueness and continuous dependence of initial conditions for linear differential equations shows that this convergence is uniform also on compacts in the right half-plane of the ζ-plane.

It remains to notice that the univalent functions $\psi_L : H \to D_L,\ \psi_L(\delta) = z_1$ also converge uniformly on compact subsets of the right half-plane as $t \to 0$. This is a consequence of the Caratheodory convergence theorem, see for example [25, Ch. 2, §5].

This proved Theorem 3.3 for the case that $K' \subset \psi_L(H)$. However the uniqueness theorem for linear differential equations implies now that the statement holds for every compact set $K' \subset \mathbf{C}$.

The integral equation (3.15) implies that $v(\zeta) \to 1$ as $\mathrm{Re}\,\zeta \to +\infty$. Going back to $y(z)$ we obtain (3.9), so indeed we constructed the Sibuya solution. □

Now we address convergence of eigenfunctions. On this we have a general result.

Theorem 3.4. *Consider the one-parametric family of eigenvalue problems (3.4), (3.6), and suppose that $\lambda(t)$ is an eigenvalue which depends continuously on t for $t \in [0, t_0]$. Then there exist eigenfunctions y_t which depend continuously on t for $t \in [0, t_0]$. Topology on eigenfunctions is of uniform convergence on compact subsets.*

Proof. Let y_t be the Sibuya solution of (3.4) with $\lambda = \lambda(t)$, corresponding to the ray L_1. It depends continuously on t by Theorem 3.3. Let y_t^* be the Sibuya solution of the same equation corresponding to L_2. The assumption that $\lambda(t)$ is an eigenvalue implies that y_t and y_t^* are proportional. This implies that y_t tends to zero on L_2, so y_t satisfies both boundary conditions. Thus y_t is an eigenfunction that depends continuously on t. □

4. PT-symmetric potentials and linear differential equations having solutions with prescribed number of non-real zeros

Hellerstein and Rossi asked the following question [9, Problem 2.71]. Let

$$w'' + Pw = 0 \tag{4.1}$$

be a linear differential equation with polynomial coefficient P. Characterize all polynomials P such that the differential equation admits a solution with infinitely many zeros, all of them real.

This problem was investigated in [40, 28, 27, 31, 36, 22]. Recently K. Shin [37] announced a description of polynomials P of degree 3 or 4 such that equation (4.1) has a solution with infinitely many zeros, *all but finitely many of them* real. It turns out that equations (4.1) with this property are equivalent to (1.4) or (1.6) of the Introduction by an affine change of the independent variable.

Here we use the methods of [21, 19] to parametrize polynomials P of degrees 3 and 4 such that equation (4.1) has a solution with *prescribed* number of non-real zeros.

We begin with degree 3.

Theorem 4.1. *For each integer $n \geq 0$ there exists a simple curve Γ_n in the plane $\mathbf{R}^2$ which is the image of a proper analytic embedding of the real line and which has the following properties.*

For every $(a, \lambda) \in \Gamma_n$ the equation

$$-w'' + (z^3 - az + \lambda)w = 0 \tag{4.2}$$

has a solution w with $2n$ non-real zeros. Real zeros belong to a ray $(-\infty, x_0)$ and there are infinitely many of them. This solution satisfies $\lim_{t\to\pm\infty} w(it) = 0$.

The union $\cup_{n=0}^{\infty}\Gamma_n$ coincides with the real part of the spectral locus of (1.4).

The projection $(a, \lambda) \mapsto a$,

$$\Gamma_n \cap \{(a, \lambda) : a \geq 0\} \to \{a : a \geq 0\}$$

is a 2-to-1 covering map. The curves Γ_n are disjoint, and for $a \geq 0$ and $n \geq 0$, if $(a, \lambda) \in \Gamma_n$ and $(a, \lambda') \in \Gamma_{n+1}$ then $\lambda < \lambda'$.

Equation (4.2) is equivalent to the PT-symmetric equation (1.4) in the Introduction by the change of the independent variable $z \mapsto iz$. Computer experiments strongly suggest that the projection $(a, \lambda) \mapsto a$ is 2-to-1 on the whole curve Γ_n except one critical point of this projection, and that the whole curve Γ_{n+1} lies above Γ_n.

Fig. 18, taken from Trinh's thesis [41] (see also [14]), shows a computer generated picture of the curves Γ_n.

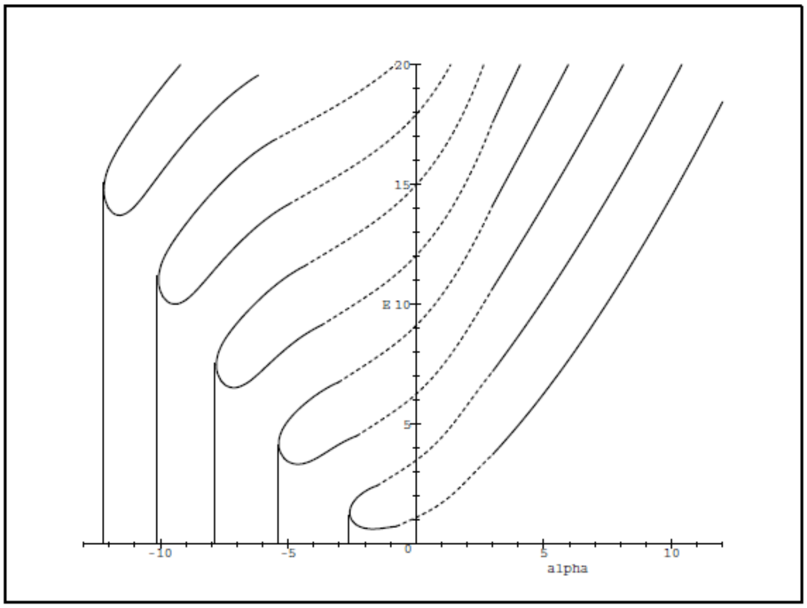

Fig 18. Curves $\Gamma_n,\ n = 0, \ldots, 4$ in the (a, λ) plane (Trinh, 2002).

Proof. Consider the Stokes sectors of equation (4.2). We enumerate them counter-clockwise as $S_0, \ldots S_4$ where S_0 is bisected by the positive real axis. Consider the set G of all real meromorphic functions f whose Schwarzian derivatives are real polynomials of the form $-2z^3+a_2z^2+a_0$, and whose asymptotic values in the sectors $S_0, \ldots, S_4$ are $\infty, 0, b, \overline{b}, 0$ respectively where $b = e^{i\beta}, \beta \in (0, \pi)$. Such functions are described by certain cell decompositions of the plane [19]. By a cell decomposition we understand a representation of a space X as a union of disjoint cells. This union is locally finite, and the boundary of each cell consists of cells of smaller dimension. The 0-cells are points, vertices of the decomposition. The 1-cells are embedded open intervals, the edges, and the 2-cells are embedded open discs, faces of a decomposition. Two cell decompositions of a space X are called equivalent if they correspond to each other via an orientation-preserving homeomorphism of X.

To describe functions of the set G, we begin with the following cell decomposition Φ of the Riemann sphere:

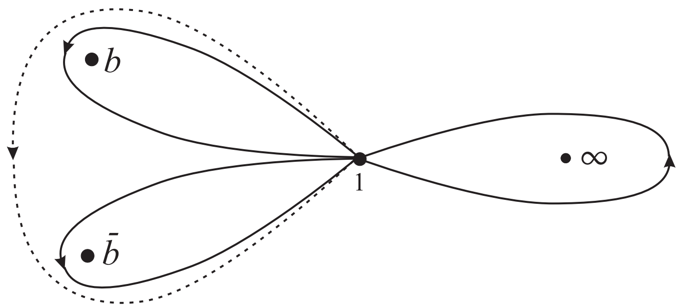

Fig 19. Cell decomposition Φ of the image sphere (solid lines).

It consists of one vertex at 1 and three edges which are simple disjoint loops around $b, \overline{b}$ and ∞, so that the loop around ∞ is symmetric with respect to complex conjugation while the loops about b and $\overline{b}$ are interchanged by the complex conjugation. The point 0 is outside of the Fig. 19. The dotted line is not discussed here; it is needed for the future.

So our cell decomposition has one vertex, three edges and four faces. The faces are labeled by points $b, \overline{b}, \infty$ and 0 which are inside the faces. (So three faces are bounded by single edge each, while one face (labeled with 0) is bounded by three edges).

Suppose now that we have a local homeomorphism $g : \mathbf{C} \to \overline{\mathbf{C}}$ such that the restriction

$$g : \mathbf{C} \backslash g^{-1}(A) \to \overline{\mathbf{C}} \backslash A, \tag{4.3}$$

where $A = \{b, \overline{b}, \infty, 0\}$, is a covering map. Then we preimage $\Psi = g^{-1}(\Phi)$ will be a cell decomposition of the plane $\mathbf{C}$. Now suppose that a cell decomposition Ψ of the plane is given in advance, and suppose that its local structure is the same as that of Φ. This means that the faces of Ψ are labeled by elements of the set A, and that a neighborhood of each vertex of Ψ can be mapped onto a neighborhood of the vertex of Φ by an orientation-preserving homeomorphism, respecting the labels of the faces. Then there exists a local homeomorphism $g : \mathbf{C} \to \overline{\mathbf{C}}$ such that (4.3) is a covering map. We use the following cell decomposition to construct g:

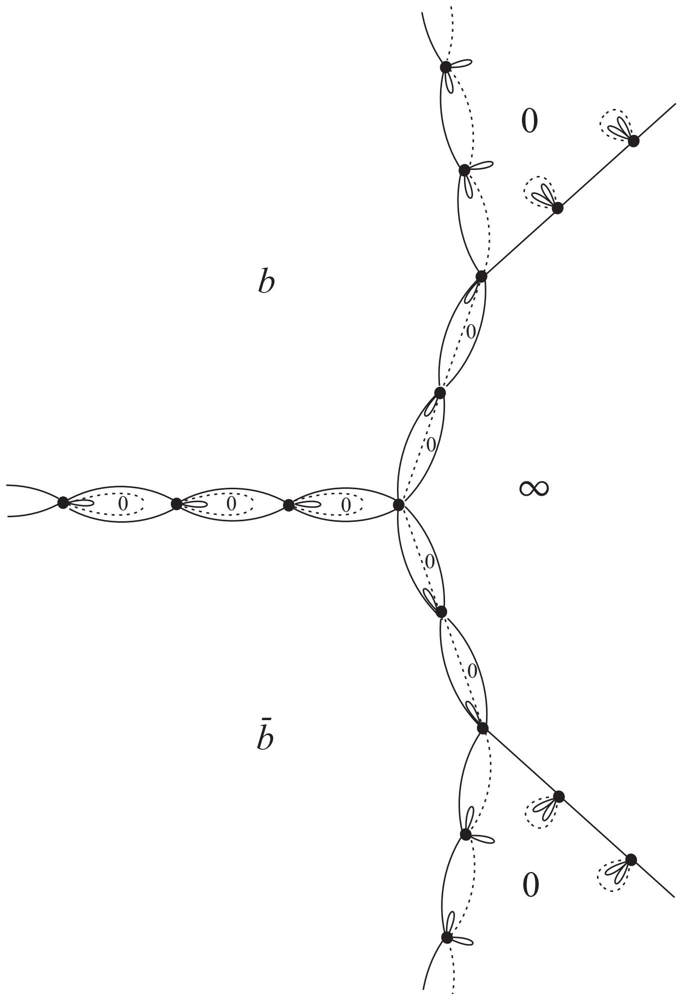

Fig 20. Cell decomposition Ψ_n for $n = 2$ (solid lines).

The five "ends" extend to infinity periodically. This cell decomposition depends on one integer parameter $n \geq 0$ which is the number of 0-labeled faces between the neighboring "ramification points". Only some face labels are shown but the reader can easily recover all other labels from the condition that in a neighborhood of each vertex Ψ_n is similar to Φ. The dotted lines are not a part of our cell decomposition; they are added for a future need. Ψ_n is symmetric with respect to the real line, this permits to make our local homeomorphism g symmetric, that is $g(\overline{z}) = \overline{g(z)}$. This construction defines the map g up to pre-composition with a symmetric homeomorphism $\phi : \mathbf{C} \to \mathbf{C}$ of the domain of g. A fundamental result of R. Nevanlinna ensures that this homeomorphism ϕ can be chosen in such a way that $f = g \circ \phi$ is a meromorphic function which is real in the sense that $f(\overline{z}) = \overline{f(z)}$. We refer to [19] for the discussion of this construction in our current context; in fact [19] contains a simple alternative proof of Nevanlinna's theorem. Nevanlinna's original proof is explained in modern language in [18]; the original paper of Nevanlinna is [34].

The meromorphic function f is defined by the cell decomposition Ψ_n and parameter b up to pre-composition with a real affine map $cz + d$. Furthermore, the Nevanlinna theory says that the Schwarzian derivative of f is a polynomial of degree exactly 3 (the number of unbounded faces of Ψ_n minus 2). We pre-compose f with a real affine map to normalize this polynomial to have leading coefficient -2 and zero coefficient at z^2. Thus

$$\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 = -2(z^3 - az + \lambda). \tag{4.4}$$

As f is real, a and λ are also real. Now f is uniquely defined by the properties that it satisfies a differential equation (4.4), has asymptotic values $0, b, \overline{b}, \infty$, and that $f^{-1}(\Phi)$ equivalent to Ψ_n (Fig. 20 for $n = 2$) by an orientation-preserving homeomorphism of the plane commuting with the reflection $z \mapsto \overline{z}$. The statement on asymptotic values implies that

$$f(it) \to 0, \quad t \to \pm\infty.$$

Furthermore, f depends analytically on b, when b is in the right half-plane, and thus we obtain a real analytic map $b \mapsto (a, \lambda)$. This map is evidently invariant with respect to transformations $b \mapsto tb,\ t \in \mathbf{R}\backslash\{0\}$, this is because the Schwarz derivative in the right hand side of (4.4) does not change when f is replaced by tf.

Thus for every n, we have a one-parametric family $G_n \subset G$ of meromorphic functions, parametrized by $\beta \in (0, \pi)$, $b = e^{i\beta}$. Taking the Schwarzian derivative we obtain a map $F_n : G_n \to \mathbf{R}^2$, $\beta \mapsto (a, \lambda)$. This map is known to be a proper real analytic immersion [2]. It is easy to see that it is injective: two solutions of the same Schwarz equation may differ only by post-composition with a fractional-linear map, and this fractional-linear map must be identity by our normalization of asymptotic values.

For the same reasons the images of F_n are disjoint: for different n, our functions have (topologically) different line complexes. The images of F_n are our curves Γ_n.

Now we prove that the union of Γ_n coincides with the real part of the spectral locus of (1.6).

Our functions $f \in G$ can be written in the form $f = w/w_1$ where w and w_1 are two linearly independent solutions of equation (4.2) with some real a and λ. We can choose w and w_1 to be real entire functions. Condition that $f(it) \to 0$ as $t \to \pm\infty$ implies that $w(it) \to 0$ for $t \to \pm\infty$ so w is an eigenfunction of the spectral problem

$$-w'' + (z^3 - az + \lambda)w = 0, \quad w(\pm i\infty) = 0, \tag{4.5}$$

which is equivalent to (1.4) by the change of the independent variable $z \mapsto iz$.

So our curves Γ_n belong to the real part of the spectral locus of (4.5) or (1.4).

Now, let λ be a real eigenvalue of the problem (4.5), w a corresponding eigenfunction. Choose a point x_0 on the real line such that $w(x_0) \neq 0$ and normalize w so that $w(x_0) = 1$. Then $w^*(z) = \overline{w(\overline{z})}$ is an eigenfunction with the same eigenvalue, so $w^* = cw$ for some constant $c \neq 0$. Substituting x_0 gives that $c = 1$. So w is real.

Let w_1 be a solution of the same equation as w but satisfying $w(x) \to 0$, $x \to +\infty$. We normalize w_1 so that w_1 is real in the same way as we normalized w. Then $f = w/w_1$ is a real meromorphic function whose Schwarzian derivative is a cubic polynomial with top coefficient -2, and the asymptotic values in S_j are $\infty, 0, b, \overline{b}, 0$. We can change the normalization of w_1 multiplying it by any real non-zero constant. In this way we achieve that $b = e^{i\beta}$ for some $\beta \in (0, \pi)$. So f belongs to the class G.

Lemma 4.2. $G = \cup_{n=0}^{\infty} G_n$.

Proof. Let $f \in G$. Consider the cell decomposition $X = f^{-1}(\Phi)$. We have to prove that $X = \Psi_n$ for some $n \geq 0$. To do this, we follow [19]. We first remove all loops from X, and then replace each multiple edge by a single edge, and denote the resulting cell decomposition by Y. Notice that the cyclic order $(\infty, b, \overline{b})$ in Fig. 19 is consistent with the cyclic order $(\infty, 0, b, \overline{b}, 0)$ of the Stokes sectors in the z-plane. By [19, Proposition 6], this implies that the 1-skeleton of Y is a tree. This infinite tree is properly embedded in the plane, has 5 faces, is symmetric with respect to the real line, and has two faces labeled with 0 which are interchanged by the symmetry. Moreover, the faces of Y are in one-to-one correspondence with the Stokes sectors, and the face corresponding to S_0 is bisected by the positive ray. One can easily classify all trees with these properties. They depend of one integer parameter $n \geq 0$ which is the distance between the ramification point in the upper half-plane and the ramification point on the real axis. Now we refer to [19, Proposition 7] that the tree Y uniquely defines the cell decomposition X. This shows that $X = \Psi_n$ for some $n \geq 0$.

Meromorphic function f is defined by the cell decomposition X and the parameter b up to an affine change of the independent variable. Normalizing it as in (4.4) gives $f \in G_n$. □

We conclude that the union of our curves Γ_n in the right half-plane $a \geq 0$ coincides with the real part of the spectral locus of (4.5).

Now we study the shape of the curves Γ_n. The boundary value problem (4.5) was considered by Shin [35], Delabaere and Trinh [41, 14]. The spectrum of this problem is discrete, simple and infinite. It is known [35] that for $a \geq 0$ all eigenvalues of this problem are real and positive. It follows from this result that there are real analytic curves $\lambda = \gamma_k(a),\ k = 0, 1, 2, \ldots$, such that for each k, $\gamma_k(a)$ is an eigenvalue of the problem (4.5), and $\gamma_k < \gamma_{k+1},\ k = 0, 1, 2, \ldots$. So the part of the real spectral locus in $\{(a, \lambda) : a \geq 0\}$ is the union of the graphs of γ_k.

Next we prove that the intersection of Γ_n with the half-plane $a \geq 0$ consists of γ_{2n} and γ_{2n+1}. For this purpose we study what happens to eigenvalues and eigenfunctions of the problem (4.5) as $a \to +\infty$.

A different approach to the asymptotics as $a \to \infty$ is used in [26]. We could use this result here instead of referring to Sections 2,3.

We substitute $cz + d$ in (4.5) and put $y(z) = w(cz + d)$, where

$$d = (a/3)^{1/2} > 0, \quad c = (3d)^{-1/4} > 0.$$

The result is

$$-y'' + (c^5 z^3 + z^2 + \mu)y = 0, \tag{4.6}$$

where $\mu = c^2(\lambda + d^3 - ad)$. Choosing the positive and negative imaginary rays as our normalization rays L_1 and L_2, we see that the sectors containing L_1 and L_2 in Fig. 9 are stable, in the sense of Definition 2.6. According to Theorem 2.21, the rays do not intersect the Stokes complex, and conditions of Theorem 3.2 are satisfied, so the spectrum of the problem (4.6) converges to the spectrum of the limit problem

$$-y'' + z^2 y = -\mu y, \quad y(\pm i\infty) = 0. \tag{4.7}$$

This limit problem is equivalent to the self-adjoint problem

$$-u'' + z^2 u = \mu u, \quad u(\pm\infty) = 0$$

by the change of the variable $u(z) = y(iz)$. Convergence of the spectrum implies convergence of eigenfunctions uniform on compact subsets of the plane by Theorem 3.4. As a varies from 0 to ∞, we can choose an eigenvalue $\lambda(a)$ which varies continuously and the corresponding eigenfunction that varies continuously, and tends to an eigenfunction of (4.6). In the process of continuous change the number of non-real zeros of the eigenfunction cannot change because eigenfunctions cannot have multiple zeros. The conclusion of the theorem will now follow from the known properties of zeros of eigenfunctions of Hermitian boundary value problems once we establish the following

Lemma 4.3. *As* $t = c^5 \to 0$ *in (4.6) the non-real zeros of an eigenfunction cannot escape to infinity.*

Notice that the real zeros of the eigenfunction do escape to infinity, as the limit eigenfunction has no real zeros except possibly 0.

Proof. Let w_t be the eigenfunction constructed in Theorem 3.4 which depends continuously on t. Let w_t^* be the Sibuya solution corresponding to the positive ray. Then f_t is a real meromorphic solution of the Schwarz equation and has asymptotic values $\infty, 0, b_t, \overline{b_t}, 0$ in the sectors S_j. As $f_t \to f_0$, and the Schwarzian of f_0 is of degree 2, we conclude that b converges to the real axis, and the Riemann surface of f_t must converge in the sense of Caratheodory [10, 43], to a Riemann surface with 4 logarithmic branch points which can lie only over $0, \infty, b_0$. To construct the cell decomposition corresponding to this limit Riemann surface we must replace in the original cell decomposition two loops corresponding to b, $\overline{b}$ with a single loop around both of these points. This loop is shown by the dotted line in Fig. 19 and its preimage is shown by the dotted lines in Fig. 20. The original loops that separate b- and $\overline{b}$-labeled faces from the face labeled 0 must be removed. Performing this operation on the cell decomposition Ψ_n we see that 1-skeleton breaks into infinitely many pieces. But there is only one piece that has four unbounded faces and thus can correspond to a meromorphic function whose Schwarzian derivative is a polynomial of degree 2. This limit decomposition is shown in Fig. 21.

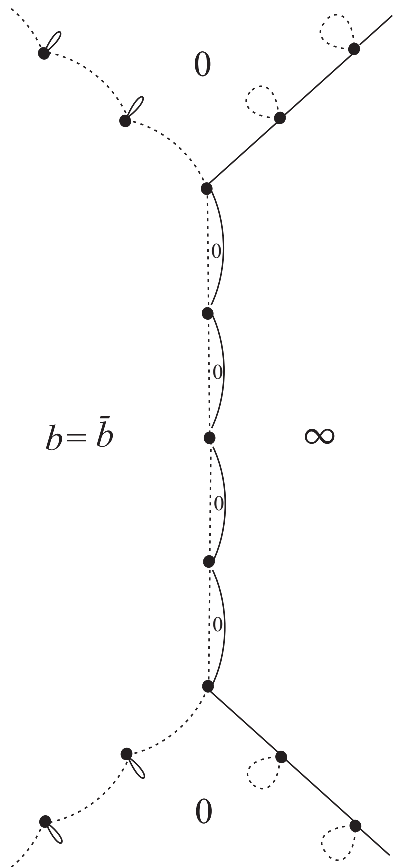

Fig 21. Limit cell decomposition with $n = 2$ (solid and dotted lines).

This time both solid and dotted lines represent the edges of this decomposition. We see that the number of non-real zeros in the limit is the same as it was before the limit. □

This completes the proof of Theorem 4.1 □

Theorem 4.4. *Let P be a polynomial of degree 3 such that equation (4.1) has a solution with $2n$ non-real zeros. Then (4.1) can be transformed to an equation of Theorem 4.1 with $(a, \lambda) \in \Gamma_n$ by a real affine change of the independent variable.*

Proof. By the results of Gundersen [27, 28], all solutions have infinitely many zeros, and the coefficients of P are real. By a real affine change of the variable we achieve that $P(z) = -z^3 + az - \lambda$. As almost all zeros are real, our solution must tend to zero in both directions of the imaginary axis. So λ is an eigenvalue of the problem (4.5). Let w be a real eigenfunction and w_1 a real solution of our equation that is linearly independent of w. Then the ratio $f = w/w_1$ is a meromorphic function which is a local homeomorphism, and has asymptotic values $\infty, 0, b, \overline{b}, 0$ in $S_0, \ldots, S_4$, respectively. After a real affine change of the independent variable, this function will belong to the class G defined in the proof of Theorem 4.1. □

Now we state analogous results for quartic oscillators. There are two different real two-parametric families in which solutions with finitely many non-real zeros can occur [37].

$$-w'' + (-z^4 + az^2 + cz + \lambda)w = 0, \quad w(\pm i\infty) = 0. \tag{4.8}$$

studied in [3, 13], and

$$-w'' + (z^4 - 2az^2 + 2mz + \lambda)w = 0, \quad w(te^{i\theta}) \to 0, \quad \theta = \pm\pi/3 \tag{4.9}$$

studied in [4]. Here $m \geq 1$ is an integer. Problem 4.9 is quasi-exactly solvable, which means that there are m eigenfunctions of the form $p(z)\exp(z^3/3 - az)$, where p is a polynomial of degree $m - 1$. The families (4.8) and (4.9) are equivalent to the PT symmetric families (1.6) and (1.7-1.8) of the Introduction via the change of the independent variable $z \mapsto iz$.

Theorem 4.5. *The real part of the spectral locus of (4.8) consists of disjoint smooth connected analytic surfaces S_n, $n \geq 0$, properly embedded in $\mathbf{R}^3$. For $(a, c, \lambda) \in S_n$, the eigenfunction has $2n$ non-real zeros. Each of these surfaces is homeomorphic to a punctured disc. Projection $\pi(a, c, \lambda) = (a, c)$ has the following properties: It is a 2-to-1 covering over some neighborhood of the a-axis, and for $a > a_0$, the preimage of every line $c = \text{const}$ is compact and homeomorphic to a circle.*

Proof. We follow the same pattern as in the proof of Theorem 4.1. There are 6 Stokes sectors, $S_0, \ldots, S_5$, which we enumerate anticlockwise, beginning from the sector in the first quadrant.

If $f = w/w_1$ where w is a real eigenfunction and w_1 is a real linearly independent solution of the same equation, then f has asymptotic values $b_0, 0, b_1, \overline{b_1}, 0, \overline{b_0}$ in the sectors $S_0, \ldots, S_5$. Here $b_0 \neq b_1$, and b_0, b_1 must belong to $\mathbf{C}\backslash\mathbf{R}$.

If $c = 0$, we can choose w, w_1 with the additional symmetry with respect to the imaginary axis, which gives $b_0 = -\overline{b_1}$, so b_0 and b_1 belong to the same half-plane of $\mathbf{C}\backslash\mathbf{R}$. The same situation persists for all real c because b_0, b_1 depend continuously on c and never cross the real line. The real affine group acts on f by post-composition; this corresponds to the change of normalization of w and w_1. So we can always choose the normalization so that $b_1 = i$.

Notice that after this normalization condition $c = 0$ corresponds to $|b_0 - i/2| = 1/2$. See Remark 4.6 after the proof.

Now consider the following cell decomposition Φ of the Riemann sphere (the range of f):

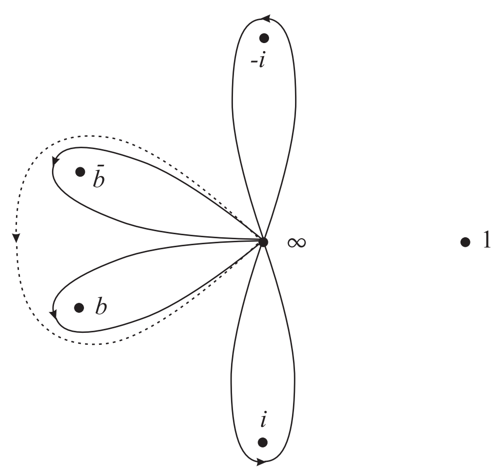

Fig 22. Cell decomposition Φ of the sphere (solid lines).

It consists of one vertex at ∞ and four disjoint loops around $\pm i$ and $b, \overline{b}$ that are interchanged by the symmetry.

Now consider the following cell decomposition Ψ_n of the plane (with labeled faces) which is locally similar to Φ.

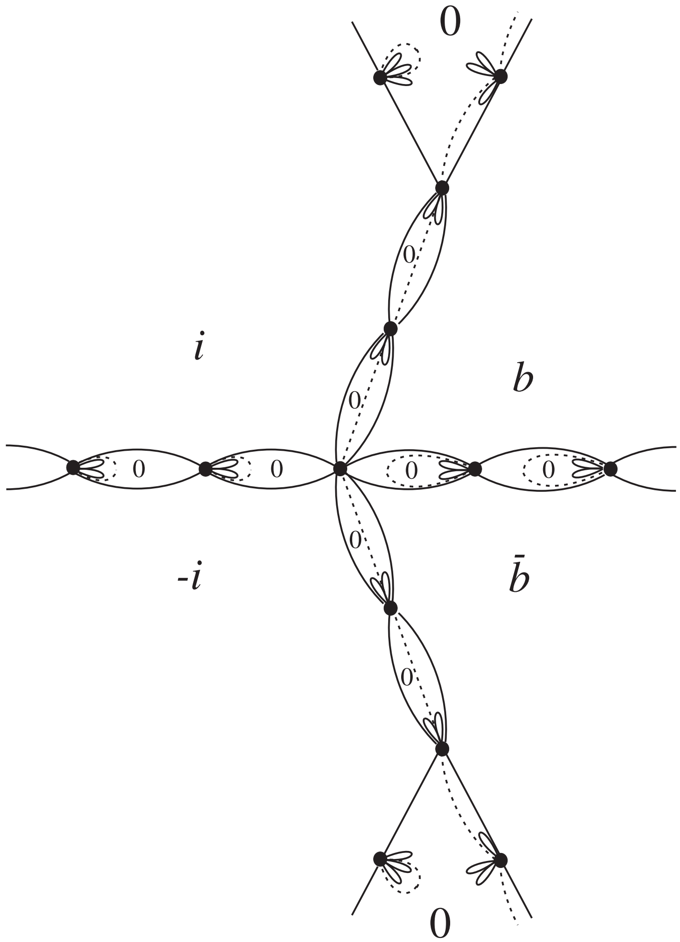

Fig 23. Cell decomposition Ψ_n for a quartic with $n = 2$ (solid lines).

The cell decomposition Ψ_n depends on one integer parameter $n \geq 0$ which is the number of 0-labeled faces between the adjacent "ramification points".

As in Theorem 4.1, Nevanlinna theory gives for each $n \geq 0$ a family G_n of meromorphic functions f which have $2n$ non-real zeros and satisfy the Schwarz equation of the form

$$\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 = 2(z^4 - az^2 - cz - \lambda). \tag{4.10}$$

with real a, c, λ.

Classification result for symmetric trees with 6 faces in [19] ensures that all equations (4.1) having a solution with infinitely many real zeros and $2n$ non-real zeros are equivalent to equations which arise from our families G_n.

This also has an implication that there is "no monodromy" in our families G_n: when b traverses a loop around i, we return with the same function f we started with. Indeed, in the process of continuous deformation the number of non-real zeros cannot change, and there is only one suitable cell decomposition Ψ_n for every n.

Thus our family G_n is homeomorphic to a punctured disc. Taking the coefficients a, c, λ of the Schwarzian defines an analytic embedding of G_n to $\mathbf{R}^3$. This is our surface S_n. The surfaces are disjoint and properly embedded for the same reasons as in the proof of Theorem 4.1.

To study the shape of these surfaces S_n in $\mathbf{R}^3$, we first notice that for $c = 0$, the eigenvalue problem obtained from (4.8) by rotation $z \mapsto iz$ is Hermitian. It follows that the intersection with the plane $S_n \cap \{(a, c, \lambda) : c = 0\}$ consists of the disjoint graphs of two analytic functions defined for all real a, and that $\lambda_n(a, 0) < \lambda_{n+1}(a, 0)$. Another simple property of the surface S_n is that it is symmetric with respect to change $c \mapsto -c$, which follows by changing $z \mapsto -z$ in the equation.

Now we study the asymptotic behavior of S_n for $a \to +\infty$. In the equation (4.8) we set $z = \varepsilon(\zeta - t),\ y(z) = w(\varepsilon(\zeta - t))$, where t satisfies

$$a - 6\varepsilon^2 t^2 = 0, \quad \text{and} \quad 4\varepsilon^6 t = 1, \tag{4.11}$$

and obtain

$$-y'' + (-\varepsilon^6 z^4 + z^3 + \alpha z + \mu)y = 0, \tag{4.12}$$

where

$$\alpha = 4\varepsilon^6 t^3 + 2\varepsilon^4 at + c\varepsilon^3,$$

and

$$\mu = -\varepsilon^6 t^4 + a\varepsilon^4 t^2 - c\varepsilon^3 t + \varepsilon^2 \lambda.$$

Expressing t and a from equations (4.11) as functions of ε and substituting the result to the expression of α we obtain

$$a = (3/8)\varepsilon^{-10}, \quad c = \varepsilon^{-3}\alpha - (1/4)\varepsilon^{-15}, \tag{4.13}$$

$$\lambda = -21 \cdot 2^{-8}\varepsilon^{-20} + (\alpha/4)\varepsilon^{-8} + \mu\varepsilon^{-2}. \tag{4.14}$$

Consider the curves Γ_n from Theorem 4.1. It follows from their properties stated in Theorem 4.1, that for every n, there exists $\alpha_n = \max\{-\alpha : (\alpha, \lambda) \in \Gamma_n$, and $0 < \alpha_n < \infty$.

Suppose that $\alpha < \alpha_n$, and consider the curve in (a, c)-plane parametrized by (4.13). Equation (4.12) satisfies the conditions of Theorem 4.1 of Section 2 (the Stokes complex corresponding to this equation is shown in Fig. 3, rotated by 90°), and the sectors containing the normalizing rays are stable. We conclude that the spectrum of (4.12) tends to the spectrum of the cubic

$$-y'' + (z^3 + \alpha z + \mu)y = 0. \tag{4.15}$$

The spectrum of the cubic with parameter α has at least one eigenvalue μ^* which is real and such that the corresponding eigenfunction has $2n$ non-real zeros. As μ^* is an isolated point of this spectrum, and the spectrum of (4.12) is symmetric with respect to the real axis, we conclude that there is an eigenvalue μ of (4.12) which is real, and the corresponding eigenfunction has $2n$ non-real zeros.

To ensure that the number of non-real zeros does not change in the limit, we make an argument similar to that in the proof of Theorem 4.1, the degeneration of the cell decomposition Ψ_n is shown in Fig. 24.

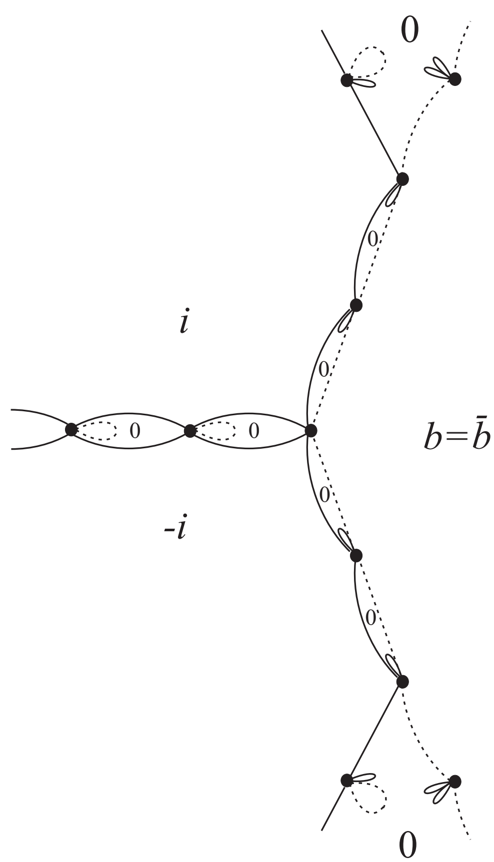

Fig 24. Limit cell decomposition with $n = 2$ (solid and dotted lines).

We conclude that projection of our surface S contains a piece of the curve (4.13) for $\varepsilon \in (0, \varepsilon_0)$.

Now suppose that $\alpha > \alpha_n$. We claim that there are no points on S_n with $a \to \infty$ and (a, c) on the curve (4.13). Proving this by contradiction, we suppose that there is a sequence $(a_j, c_j, \lambda_j) \in S_n$ such that (a_j, c_j) belong to the curve (4.13). Then Theorem 3.2 implies that the sequence μ_j related to the λ_j by (4.14), has the property that μ_j tends to a real eigenvalue μ^* of the cubic oscillator (4.15). Then the corresponding eigenfunction tends to an eigenfunction of the cubic with $2n$ non-real zeros. This is a contradiction because $\alpha > \alpha_n$, so our claim is proved.

So the projection of S_n on the plane (a, c) looks as a paraboloid $9c^2 - 4a^3 \leq 0$ when $a \to +\infty$. □

Figure 25, which is taken from Trinh's thesis shows a section of the surfaces S_n by the plane $a = -9$. Similar pictures can be seen in [3, 13].

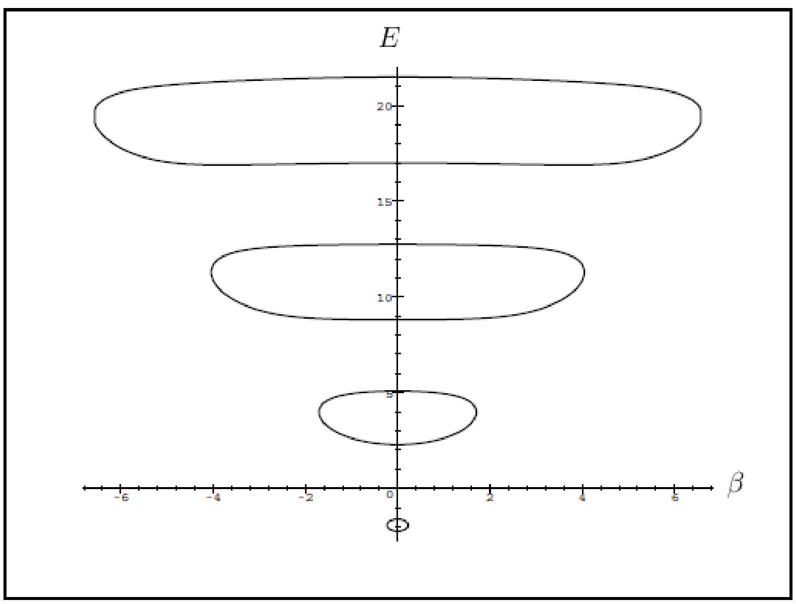

Fig. 25. Section of the surfaces $S_n,\ n = 0, 1, 2, 3$ by the plane $a = -9$.

Computational evidence suggests that each S_n has the shape of an infinite funnel with a sharp end stretching towards $a = -\infty$. This end probably corresponds to $b \to i$, where b is the asymptotic value as in Figs. 22–23. $\lambda \to -\infty$ as $a \to -\infty$ on S_n as the picture in [13] suggests. For every real a_0 the section of S_n by the plane $a = a_0$ is an oval that projects on the c-axis 2-to-1. We only proved that this section is compact for a large enough. For $n = 0, 1, 2, \dots$, the funnels S_n are symmetric with respect to $c \mapsto -c$, S_{n+1} lies above S_n and S_{n+1} is wider than S_n.

Remark 4.6. In general, it is hard to say anything explicit on the correspondence between the parameters a, c in the potential and Nevanlinna parameter b. Some information on this correspondence can be extracted from symmetry and degeneration considerations. In the beginning of the proof of Theorem we noticed that the line $c = 0$ corresponds to the circle $|b - i/2| = 1/2$. We can determine now the sign of c for b inside and outside of this circle. Degeneration used in the proof corresponds to convergence of b to a real non-zero point. Formula (4.13) shows that $c < 0$ when $\epsilon \to 0$. So negative c correspond to the exterior of the circle and positive c to its interior.

Now we state the result about the second PT-symmetric family of quartics.

Theorem 4.7. *The real QES part of the spectral locus of (4.9) consists of* $\lceil m/2 \rceil$ *simple disjoint analytic curves* $\Gamma^*_{m,n}$, $n = 0, \dots, \lceil m/2 \rceil - 1$, *properly embedded curves which for* $a > 0$ *project onto the ray* $a > 0$ 2-*to* 1. *When* $(a, \lambda) \in \Gamma^*_n$, *the eigenfunction has* $2n$ *non-real zeros.*

The proof is completely similar to the proof of Theorem 4.1.

The problem of study of the whole real part of spectral locus of (4.9), as a two-parametric family with real m seems quite interesting and challenging. A picture of the spectral locus for $m = 3$ can be seen in [4].

Theorem 4.8. *Every equation of the form (4.1) with polynomial* P *of degree* 4 *which has a solution with* $2n$ *non-real zeros and infinitely many real zeros is equivalent by a real affine change of the independent variable to equation (4.8) with* $(a, c, \lambda) \in S_n$.

Every equation of the form (4.1) with polynomial P *of degree* 4 *which has a solution with finitely many zeros is equivalent to (4.9) with* $(a, \lambda) \in \Gamma^*_{m,n}$ *by an affine change of the independent variable.*

The proof is similar to that of Theorem 4.4.

References

[1] P. Alexandersson and A. Gabrielov, On eigenvalues of the Schrödinger operator with a polynomial potential with complex coefficients, in preparation.

[2] I. Bakken, A multiparameter eigenvalue problem in the complex plane. Amer. J. Math. 99 (1977), no. 5, 1015–1044.

[3] C. Bender, M. Berry, P. Meisinger, Van M. Savage, M. Simsek, Complex WKB analysis of energy-level degeneracies of non-Hermitian Hamiltonians. J. Phys. A 34 (2001), no. 6, L31–L36.
[4] C. Bender, S. Boettcher, Quasi-exactly solvable quartic potential. J. Phys. A 31 (1998), no. 14, L273–L277.
[5] C. Bender and S. Boettcher, Real spectra in non-Hermitian Hamiltonians having PT symmetry, Phys. Rev. Lett., 80 (1998) 5243–5246.
[6] C. Bender, S. Bottcher, V. M. Savage, Conjecture on the interlacing of zeros in complex Sturm-Liouville problems, J. Math. Phys. 41 (2000), no. 9, 6381–6387.
[7] C. Bender, A. Turbiner, Analytic continuation of eigenvalue problems. Phys. Lett. A 173 (1993), no. 6, 442–446.
[8] C. Bender and Tai Tsun Wu, Anharmonic oscillator. Phys. Rev. (2) 184 1969 1231–1260.
[9] D. Brannan and W. Hayman, Research problems in complex analysis, Bull. London Math. Soc., 21 (1989) 1–35.
[10] C. Carathéodory, Untersuchungen über die konformen Abbildungen von festen und veränderlichen Gebieten, Math. Ann., 72 (1912) 107–144.
[11] E. Caliceti, S. Graffi and M. Maioli, Perturbation theory of odd anharmonic oscillators, Comm. Math. Phys., 75 (1980) 51–66.
[12] E. Delabaere, F. Pham, Eigenvalues of complex Hamiltonians with PT-symmetry. I. Phys. Lett. A 250 (1998), no. 1-3, 25–28.
[13] E. Delabaere, F. Pham, Eigenvalues of complex Hamiltonians with PT-symmetry. II. Phys. Lett. A 250 (1998), no. 1-3, 29–32.
[14] E. Delabaere, Duc Tai Trinh Spectral analysis of the complex cubic oscillator. J. Phys. A 33 (2000), no. 48, 8771–8796.
[15] P. Dorey, C. Dunning and R. Tateo, Spectral equivalences, Bethe ansatz equations, and reality properties in PT-symmetric quantum mechanics, J. Phys. A, 34 (2001) 5679–5704.
[16] P. Dorey, C. Dunning and R. Tateo, A reality proof in PT-symmetric quantum mechanics, Czech. J. Phys., 54 (2004) 5.
[17] P. Dorey, C. Dunning and R. Tateo, The ODE/IM correspondence arXiv:/hep-th/0703066v2.
[18] A. Eremenko, Geometric theory of meromorphic functions, in the book: "In the tradition of Ahlfors and Bers III", Contemp. math. 355, AMS, Providence, RI, 2004. Expanded version: www.math.purdue.edu/~eremenko/dvi/mich.pdf
[19] A. Eremenko, A. Gabrielov, Analytic continuation of eigenvalues of a quartic oscillator, Comm. Math. Phys., v. 287, No. 2 (2009) 431-457.
[20] A. Eremenko and A. Gabrielov, Irreducibility of some spectral determinants, arXiv:0904.1714.
[21] A. Eremenko A. Gabrielov and B. Shapiro, Zeros of eigenfunctions of some anharmonic oscillators, Ann. Inst. Fourier, Grenoble, 58, 2 (2008) 603-624.
[22] A. Eremenko and S. Merenkov, Nevanlinna functions with real zeros, Illinois J. Math., 49, 3-4 (2005) 1093–1110.
[23] M. A. Evgrafov and M. V. Fedorjuk, Asymptotic behavior of solutions of the equation $w''(z) - p(z, \lambda)w(z) = 0$ as $\lambda \to \infty$ in the complex z-plane. (Russian) Uspehi Mat. Nauk 21 1966 no. 1 (127), 3–50.
[24] M. V. Fedoryuk, Asymptotic analysis. Linear ordinary differential equations. Springer-Verlag, Berlin, 1993.

[25] G. M. Goluzin, Geometric theory of functions of a complex variable, AMS, Providence, RI, 1969.
[26] V. Grecchi, M. Maioli and A. Martinez, Padé summability of the cubic oscillator, J. Math. Phys. A: Mqth. Theor. 42 (2009) 425208.
[27] G. Gundersen, On the real zeros of solutions of $f'' + A(z)f = 0$ where A is entire, Ann. Acad. Sci. Fenn. Math. 11, 1986 276–294.
[28] G. Gundersen, SolutionS of $f'' + P(z)f = 0$ that have almost all real zeros, Ann. Acad. Sci. Fenn. Math., 26 (2001) 483–488.
[29] C. Handy, Generating converging bounds to the (complex) discrete states of the $P^2 + iX^3 + i\alpha X$ Hamiltonian, J. Phys. A: Math. Gen. 34 (2001) 5065–5081.
[30] J. Heading, An introduction to phase-integral methods, Methuen, London; John Wiley, NY 1962.
[31] S. Hellerstein and J. Rossi, Zeros of meromorphic solutions of second order linear differential equations, Math. Z., 192 (1986) 603–612.
[32] E. Hille, Lectures on ordinary differential equations. Addison-Wesley Publ. Co., Reading, Mass.-London-Don Mills, Ont. 1969
[33] D. Masoero, Poles of *intégrale tritronqueée* and anharmonic oscillators. A WKB approach, arXiv:0909.5537v2, 2010.
[34] R. Nevanlinna, Über Riemannsche Flächen mit endich vielen Windungspunkten, Acta Math., 58 (1932) 295–373.
[35] K. Shin, On the reality of the eigenvalues for a class of PT-symmetric oscillators, Comm. Math. Phys., 229 (2002) 543–564.
[36] K. Shin, New polynomials P for which $f'' + P(z)f = 0$ has a solution with almost all real zeros. Ann. Acad. Sci. Fenn. Math. 27 (2002), no. 2, 491–498.
[37] K. Shin, All cubic and quartic polynomials P for which $f'' + P(z)f = 0$ has a solution with infinitely many real zeros and at most finitely many non-real zeros, Abstracts AMS 1057-34-26 (Lexington, KY, March 27-28, 2010).
[38] Y. Sibuya, Global theory of a second order linear ordinary differential equation with a polynomial coefficient, North-Holland, Amsterdam–Oxford; American Elsevier, NY, 1975.
[39] B. Simon, Coupling constant analyticity for the anharmonic oscillator, Ann. Physics 58 (1970), 76–136.
[40] E. Titchmarsh, Eigenfunction expansions associated with second order differential equations, vol. 1, second edition, Oxford, Oxford UP, 1962.
[41] Duc Tai Trinh, Asymptotique et analyse spectrale de l'oscillateur cubique, These, 2002.
[42] Duc Tai Trinh, On the Sturm–Liouville problem for the complex cubic oscillator, Asymptot. Anal. 40 (2004) 211-234.
[43] L. Volkovyski, Converging sequences of Riemann surfaces, Mat. Sbornik, 23(65) (1948), N 3, 361–382. (Russian). Oxford UP, 1989.

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