SINGULAR PERTURBATION OF POLYNOMIAL POTENTIALS WITH APPLICATIONS TO PT-SYMMETRIC FAMILIES

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ABSTRACT. We discuss eigenvalue problems of the form $-w'' + Pw = \lambda w$ with complex polynomial potential $P(z) = tz^d + \ldots$, where t is a parameter, with zero boundary conditions at infinity on two rays in the complex plane. In the first part of the paper we give sufficient conditions for continuity of the spectrum at $t = 0$. In the second part we apply these results to the study of topology and geometry of the real spectral loci of PT-symmetric families with P of degree 3 and 4, and prove several related results on the location of zeros of their eigenfunctions.

MSC: 34M35, 35J10. Keywords: singular perturbation, one-dimensional Schrödinger operators, eigenvalue, spectral determinant, PT-symmetry.

1. Introduction

We consider eigenvalue problems

$$(1.1) \qquad -w'' + P(z, \mathbf{a})w = \lambda w, \quad y(z) \to 0 \text{ as } z \to \infty,\ z \in L_1 \cup L_2.$$

Here P is a polynomial in the independent variable z, which depends on a parameter $\mathbf{a}$, and L_1, L_2 are two rays in the complex plane. The set of all pairs $(\mathbf{a}, \lambda)$ such that λ is an eigenvalue of (1.1) is called the *spectral locus*.

Such eigenvalue problems were considered for the first time in full generality by Sibuya [40] and Bakken [2]. Sibuya proved that under certain conditions on L_1, L_2 and the leading coefficient of P, there exists an infinite sequence of eigenvalues tending to infinity. If

$$(1.2) \qquad P(z, \mathbf{a}) = z^d + a_{d-1}z^{d-1} + \ldots + a_1 z,$$

where $\mathbf{a} = (a_1, ..., a_{d-1})$, then the spectral locus, which is the set of all $(\mathbf{a}, \lambda) \in \mathbf{C}^d$ such that λ is an eigenvalue of 1.1, is described by an equation $F(\mathbf{a}, \lambda) = 0$. Here $F(\mathbf{a}, \lambda)$ is an entire function of d variables,

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called the *spectral determinant.* So the spectral locus of (1.1), (1.2) is an analytic hypersurface in $\mathbf{C}^d$. It is smooth [2] and connected for $d \geq 3$ [1, 30].

In the first part of this paper we study what happens to the eigenvalues and eigenfunctions when the leading coefficient of P tends to zero.

Bender and Wu [8] studied the quartic oscillator as a perturbation of the harmonic oscillator:

$$-w'' + (\varepsilon z^4 + z^2)w = \lambda w, \quad w(\pm\infty) = 0. \tag{1.3}$$

Here and in what follows $w(\pm\infty) = 0$ means that the boundary conditions are imposed on the positive and negative rays of the real line. It has been known for long time that the eigenvalues of (1.3) converge as $\varepsilon \to 0+$ to the eigenvalues of the same problem with $\varepsilon = 0$, but they are not analytic functions of ε at $\varepsilon = 0$ (perturbation series diverge). To investigate this phenomenon, Bender and Wu considered complex values of ε and studied the analytic continuation of the eigenvalues as functions of ε in the complex plane. Their main findings can be stated as follows: the spectral locus of the problem (1.3) consists of exactly two connected components; for $\varepsilon \neq 0$, the only singularities of eigenvalues as functions of ε are algebraic branch points. These statements were rigorously proved in [19]. Discoveries of Bender and Wu generated large literature in physics and mathematics. For a comprehensive exposition of the early rigorous results we refer to [41].

To perform analytic continuation of eigenvalues of (1.3) and similar problems for complex parameters, one has to rotate the normalization rays where the boundary conditions are imposed. One of the early papers in the physics literature that emphasized this point was [7]. Thus physicists were led to problem (1.1), previously studied only for its intrinsic mathematical interest.

An interesting phenomenon was discovered by Bessis and Zinn-Justin. For the boundary value problem

$$-w'' + iz^3 w = \lambda w, \quad w(\pm\infty) = 0,$$

they found by numerical computation that the spectrum is real. This is called the Bessis and Zinn-Justin conjecture (see, for example, historical remark in [5]). This conjecture was later proved by Dorey, Dunning and Tateo [15, 16] with a remarkable argument which they call the ODE-IM correspondence, see their survey [17]. Shin [37] extended this result to potentials

$$-w'' + (iz^3 + iaz)w = \lambda w, \quad w(\pm\infty) = 0, \tag{1.4}$$

with $a \geq 0$.

These results and conjectures generated extensive research on the so-called PT-symmetric boundary value problems. PT-symmetry means a symmetry of the potential and of the boundary conditions with respect to the reflection in the imaginary line $z \mapsto -\overline{z}$. PT stands for "parity and time reversal".

It turns out that the spectral determinant of a PT-symmetric problem is a real entire function of λ, so the set of eigenvalues is invariant under complex conjugation. In contrast to Hermitian problems where the eigenvalues are always real, the eigenvalues of a PT-symmetric problem can be real for some values of parameters, but for other values of parameters some eigenvalues may be complex. So we can see the "level crossing" (collision of real eigenvalues) in real analytic families of PT-symmetric operators, the phenomenon which is impossible in the families of Hermitian differential operators with polynomial coefficients.

In this paper, we first consider the general problem (1.1) and the limit behavior of its eigenvalues and eigenfunctions when

$$P(z) = tz^d + a_m z^m + p(z), \tag{1.5}$$

with $d > m > \deg p$, as $t \to 0$, while the coefficients of p are restricted to a compact set and a_m does not approach zero. Then we apply our general results to certain families of PT-symmetric potentials of degrees 3 and 4, and prove some conjectures made by several authors on the basis of numerical evidence.

In particular, our results for the PT-symmetric cubic (1.4) imply that no eigenvalue can be analytically continued along the negative a-axis, and the obstacle to this continuation is a branch point where eigenvalues collide.

Another result is the correspondence between the natural ordering of real eigenvalues of (1.4) for $a \geq 0$ and the number of zeros of eigenfunctions that do not lie on the PT-symmetry axis, conjectured by Trinh in [44]. This correspondence is similar to that given by the Sturm–Liouville theory for Hermitian boundary value problems.

A different approach to counting zeros of eigenfunctions is proposed in [27], where the authors prove that for a large enough, the n-th eigenfunction has n zeros in a certain explicitly described region in the complex plane.

The plan of the paper is the following. In Section 2 we prove a general theorem on the continuity of discrete spectrum at $t = 0$ for potentials of the form (1.5), with boundary conditions on two given rays. Previously such problems were studied using the perturbation theory of linear operators in [26, 41, 11]. Our method is different, it is based on analytic theory of differential equations.

Verification of conditions of our general result in Section 2 is non-trivial, and we dedicate the entire Section 3 to this. The question is reduced to the study of Stokes complexes of binomial potentials $Q(z) = tz^d + cz^m,\ d > m$, which is a problem of independent interest, so we include more detail than it is necessary for our applications. The Stokes complex is the union of curves, starting at the zeros of Q, on which $Q(z)\,dz^2 < 0$, so they are vertical trajectories of a quadratic differential. Stokes complexes occur in many questions about asymptotic behavior of solutions of equations (1.1). Our study permits us to make conclusions on the behavior, as $t \to 0$, of the Stokes complexes of potentials $P(z) = tz^d + a_m(t)z^m + p_t(z)$ where $a_m(t) \to c \neq 0$ and p_t is a family of polynomials of degree $m-1$ with bounded coefficients. We mention here [35] where a topological classification of Stokes complexes for polynomials of degree 3 is given.

In the rest of the paper we apply these results to problems with PT-symmetry. In Section 4, we consider the PT-symmetric cubic family (1.4) with real a and λ. We prove that the intersection of the spectral locus with the real (a, λ)-plane consists of disjoint non-singular analytic curves $\Gamma_n,\ n \geq 0$, the fact previously known from numerical computation [14, 43, 31]. Moreover, we prove that the eigenfunctions corresponding to $(a, \lambda) \in \Gamma_n$ have exactly $2n$ zeros outside the imaginary line. (They have infinitely many zeros on the imaginary line). Furthermore, using the result of Shin on reality of eigenvalues, we study the shape and relative location of these curves Γ_n in the (a, λ)-plane and show that $a \to +\infty$ on both ends of Γ_n, and that for $a \geq 0$, Γ_n consists of graphs of two functions, that lie below the graphs of functions constituting Γ_{n+1}.

This gives PT–analog of the familiar fact for Hermitian boundary value problems that "n-th eigenfunction has n real zeros"; in our case we count zeros belonging to a certain well-defined set in the complex plane. This result proves rigorously what can be seen in numerical computations of zeros of eigenfunctions by Bender, Boettcher and Savage [6].

The result of Section 4 also gives a contribution to a problem raised by Hellerstein and Rossi [9]: describe the differential equations

$$y'' + Py = 0 \tag{1.6}$$

with polynomial coefficient P which have a solution whose all zeros are real. For polynomials of degree 3, all such equations are parametrized by our curve Γ_0, and equations having solutions with exactly $2n$ non-real zeros are parametrized by Γ_n.

The arguments in Section 4 use our parametrization of the spectral loci from [21, 19] combined with the singular perturbation results of Sections 2 and 3. These perturbation results allow us to degenerate the cubic potential to a quadratic one (harmonic oscillator) and to make topological conclusions based on the ordinary Sturm-Liouville theory.

Next we apply similar methods to two families of PT-symmetric quartics

$$-w'' + (z^4 + az^2 + icz)w = \lambda w, \quad w(\pm\infty) = 0. \tag{1.7}$$

and

$$w'' + (z^4 + 2az^2 + 2imz + \lambda)w = 0, \tag{1.8}$$

$$w(re^{\theta}) \to 0, \text{ as } r \to \infty, \ \theta \in \{-\pi/6, -5\pi/6\}, \tag{1.9}$$

where $m \geq 1$ is an integer. The first family was considered in [3] and [12, 13]. We prove that the spectral locus in the real (a, c, λ)-space $\mathbf{R}^3$ consists of infinitely many smooth analytic surfaces $S_n, \ n \geq 0$, each homeomorphic to a punctured disc, and that an eigenfunction corresponding to a point $(a, c, \lambda) \in S_n$ has exactly $2n$ zeros which do not lie on the imaginary axis. We study the shape and position of these surfaces by degenerating the quartic potential to the previously studied PT-symmetric cubic oscillator.

The second quartic family (1.8-1.9) was introduced by Bender and Boettcher [4]. It is quasi-exactly solvable (QES) in the sense that for every integer $m \geq 1$ in the potential, there are m "elementary" eigenfunctions, each having $m - 1$ zeros. The part Z_m of the spectral locus corresponding to these elementary eigenfunctions is a smooth connected curve in $\mathbf{C}^2$ [19, 20]. In the end of Section 4 we study the intersection of this curve with the real (a, λ)-plane. Similarly to the case of the PT-symmetric cubic, this intersection consists of smooth analytic curves $\Gamma^*_{m,n}$, $n = 0, \ldots, \lceil m/2 \rceil$, and for $(a, \lambda) \in \Gamma^*_{m,n}$ the eigenfunction has exactly $2n$ zeros that do not lie on the imaginary axis. For $n \leq m/2$ the part of $\Gamma^*_{m,n}$ over some ray $a > a_m$ consists of disjoint graphs of two functions, and we have the following ordering: $(a, \lambda) \in \Gamma^*_{m,n}$, $(a, \lambda') \in \Gamma^*_{m,n+1}$ and $a > a_m$ imply that $\lambda' > \lambda$. Moreover, the QES spectrum for $a > a_m$ consists of the m smallest real eigenvalues.

The results of Section 4 permit us to answer the question of Hellerstein and Rossi stated above for polynomial potentials of degree 4: All equations (1.6) that possess a solution with $2n$ non-real zeros are parametrized by our curves $\Gamma^*_{m,n}$ if the total number of zeros is $m - 1$, and by our surfaces S_n if the total number of zeros is infinite.

Notation and conventions.

1. What we call Stokes lines is called by some authors "anti-Stokes lines" and vice versa. We follow terminology of Evgrafov and Fedoryuk [23, 24].

2. We prefer to replace z by iz in PT-symmetric problems. Then potentials become real, and the difference between PT-symmetric and self-adjoint problems is that in PT-symmetric problems the complex conjugation *interchanges* the two boundary conditions, while in self-adjoint problems both boundary conditions remain fixed by the symmetry. The main advantage for us in this change of the variable is linguistic: we frequently refer to "non-real" zeros. The expression "non-real" excludes 0, while the expression "non-imaginary" does not.

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2. Perturbation of eigenvalues and eigenfunctions

We begin with recalling some facts about boundary value problem (1.1) with potential (1.2). The *separation rays* are defined by

$$\operatorname{Re}\left(\int_0^z \sqrt{\zeta^d}\, d\zeta\right) = 0, \quad \text{that is} \quad z^{d+2} < 0.$$

These rays divide the plane into $d+2$ open sectors S_j which we call *Stokes sectors*. We enumerate them by residues modulo $d+2$ counter-clockwise.

A solution w of the differential equation (1.1) is called *subdominant* in S_j if $w(rz) \to 0$ as $r \to +\infty$, for all $z \in S_j$. For every j, the space of solutions of the equation in (1.1) which are subdominant in S_j is one-dimensional. If S_j and S_k are *adjacent*, that is $j = k \pm 1 \bmod (d+2)$ then the corresponding subdominant solutions are linearly independent.

Let S_j and S_k be two non-adjacent Stokes sectors. We consider the boundary conditions

$$w \text{ is subdominant in } S_j \text{ and } S_k. \tag{2.1}$$

Such boundary value problem has an infinite set of eigenvalues tending to infinity. All eigenspaces are one-dimensional. These facts were proved by Sibuya [40] whose main tool were special solutions normalized on one ray, which we call *Sibuya solutions*. Precise definition is given below. Our first goal is to prove continuous dependence of Sibuya solutions on parameters.

We consider a family of polynomial potentials with parameters $(t, \mathbf{a})$:

$$Q(z, t, \mathbf{a}) = tz^d + \sum_{j=0}^{m} a_j z^j, \quad m < d, \quad \mathbf{a} = (a_0, \ldots, a_m). \tag{2.2}$$

Let $K \subset \mathbf{C}^{m+1}$ be a compact set which has a fundamental system of open simply connected neighborhoods, and such that $a_m \neq 0$ for $\mathbf{a} \in K$. This compact K will be fixed in all our arguments, so our notation does not reflect dependence of the quantities introduced below on K. Let $L = \{te^{i\theta} : t \geq t_0\}$ be a ray in $\mathbf{C}$.

Suppose that for some $\delta > 0$ and for all $(t, \mathbf{a}) \in [0, 1] \times K$ the following conditions hold:

a) There exists $R > 0$ such that $|\arg z - \theta| \geq \delta$ for all zeros z of $Q(z, t, \mathbf{a})$ such that $|z| > R$.

b) Whenever L intersects a vertical trajectory of $Q(z, t, \mathbf{a})dz^2$ at a point z, $|z| > R$, the smallest angle between this trajectory and L at the intersection point is at least δ.

c) θ is not a separating direction of $Q(z, t, \mathbf{a})dz^2$.

One can easily show that b) implies c). Condition c) simply means that L is neither a Stokes line for $tz^d sz^2$, nor a Stokes line for $a_m z^m dz^2$.

Condition a) implies that there is a branch $Q_L^{1/4}$ of $Q^{1/4}$ analytic on $[0, 1] \times K \times L$. Let $Q_L^{1/2} = (Q_L^{1/4})^2$ be the corresponding branch of $Q^{1/2}$. We choose the original branch $Q_L^{1/4}$ in such a way that

$$\operatorname{Re} Q_L^{1/2}(z)dz \to +\infty, \quad z \to \infty, \quad z \in L.$$

This is possible in view of condition c).

Let us say that $y = y_L(z, t, \mathbf{a})$ is a Sibuya solution of

$$-y'' + Q(z, t, \mathbf{a})y = 0$$

if

$$y(z) \sim Q_L^{-1/4}(z) \exp\left(- \int_{z_0}^{z} Q_L^{1/2}(\zeta)d\zeta \right), \quad z \to \infty, \quad z \in L.$$

Here $z_0 = t_0 e^{i\theta}$. Notice that a change of t_0, results is multiplying the Sibuya solution by a factor that depends only on t and $\mathbf{a}$ but not on z.

Theorem 2.1. *Under the conditions* a), b), c) *above, there exists a unique Sibuya solution. It is an analytic function of* $(z, t, \mathbf{a})$ *in a neighborhood of* $\mathbf{C} \times (0, 1] \times K$*, continuous on* $K_1 \times [0, 1] \times K$ *for every compact* $K_1 \subset \mathbf{C}_z$.

Proof. Let

$$\phi(z)=\int_{z_0}^{z} Q_L^{1/2}(\zeta)d\zeta,$$

where the integral is taken along L This is an analytic function which maps L onto a curve γ. Condition b) implies that γ is a graph of a function (intersects every vertical line at most once), the slope of γ is bounded, and ϕ maps bijectively some neighborhood of L onto a neighborhood of γ.

Let ψ be the inverse function to ϕ.

Setting $u=(Q_L^{1/4}\circ\psi)y\circ\psi$, we obtain the differential equation

$$u''=(1-g(\zeta))u,$$

where the primes stand for differentiation with respect to ζ, and

$$g(\zeta)=\left(\frac{5}{16}\frac{Q'^2}{Q^3}-\frac{1}{4}\frac{Q''}{Q^2}\right)\circ\psi(\zeta).$$

This is equivalent to the integral equation

$$u(\zeta)=e^{-\zeta}+\int_{\zeta}^{\infty}\left(e^{\zeta-\eta}-e^{-\zeta+\eta}\right)g(\eta)u(\eta)d\eta, \tag{2.3}$$

where the path of integration is the part of γ from ζ to ∞. The integral equation is solved by successive approximation. We set $u(\zeta)=v(\zeta)e^{\zeta}$, and obtain

$$v(\zeta)=1+\frac{1}{2}\int_{\zeta}^{\infty}\left(e^{2(\zeta-\eta)}-1\right)g(\eta)v(\eta)d\eta, \tag{2.4}$$

which we abbreviate as $v=1+F(v)$. Setting $v_0=0$ and $v_{n+1}=1+F(v_n)$, we obtain

$$\|v_{n+1}-v_n\|_\infty\leq\|v_n-v_{n-1}\|\int_{\zeta}^{\infty}|g(t)|dt.$$

Here we used the fact that $\Re(\zeta-\eta)<0$ on the curve of integration because γ is a graph of a function. Now, if $\zeta>0$ is large enough, we have

$$\int_{\zeta}^{\infty}|g(\eta)|d\eta<1/2 \tag{2.5}$$

for all values of parameters $t\in[0,t_0],\mathbf{a}\in K$. We state this as a lemma:

Lemma 2.2. *There exists $b\in L$ such that for the piece L_b of L from b to ∞ and for all $(t,\mathbf{a})\in[0,1]\times K$ we have*

$$\left|\int_{L_b}\left(\frac{5}{16}\left(\frac{Q'}{Q}\right)^2-\frac{1}{4}\frac{Q''}{Q}\right)Q^{-1/2}dz\right|<1/2.$$

The integral in this lemma equals to the integral in (2.5) by the change of the variable $z = \psi(\zeta), \sqrt{Q}dz = d\zeta$.

Proof. Let $z_1, \ldots, z_d$ be all zeros of Q_t listed with multiplicity, in an order of non-decreasing moduli. Suppose that $z_1, \ldots, z_M$ are in the disc $|z| < R$ while the rest are outside. Here R is the number from condition a). We have

$$\frac{Q'}{Q} = \sum_{k=1}^{M} \frac{1}{z - z_k} + \sum_{k=M+1}^{d} \frac{1}{z - z_k} = \sigma_1(z) + \sigma_2(z),$$

and

$$\frac{Q''}{Q} = \left(\frac{Q'}{Q}\right)' + \left(\frac{Q_t'}{Q_t}\right)^2 = \sigma_1'(z) + \sigma_2'(z) + \sigma_1^2(z) + 2\sigma_1(z)\sigma_2(z) + \sigma_2^2(z).$$

First we estimate

$$|\sigma_1| \leq \frac{M}{|z| - R}, \quad |\sigma_1'(z)| \leq \frac{M}{(|z| - R)^2}. \tag{2.6}$$

To estimate σ_2 we first use condition a) to conclude that

$$|z - z_k| \geq C_0|z|, \quad z \in L, \quad k \geq M \tag{2.7}$$

where C_0 depends only on δ. Then

$$|\sigma_2(z)| \leq C_1/|z|, \quad |\sigma_2'| \leq C_2/|z|^2. \tag{2.8}$$

Applying these inequalities, we obtain that

$$\left| \frac{5}{16} \frac{Q_t'^2}{Q_t^2} - \frac{1}{4} \frac{Q_t''}{Q_t} \right| \leq \frac{C}{|z|^2},$$

on L, where C depends only on δ.

Now we write

$$|Q_t(z)| = t \prod_{j=1}^{M} |z - z_j| \prod_{j=M+1}^{d} |z - z_j| \geq t C_3^d (|z| - R|)^M \prod_{j=M+1}^{d} |z_j|.$$

Here we used inequality (2.7) with interchanged z and z_k. It is easy to see by Vieta's theorem that

$$\prod_{j=M+1}^{d} |z_j| \geq C_4 t^{-1},$$

where C_4 depends only on K. This shows that $|Q_t(z)| \geq C_6|z|^M$.

Now we use the fact that L_b is a part of a ray from the origin, so $|dz| = d|z|$ on L_b Putting all this together we conclude that our integral is majorized by the integral

$$\int_{|b|}^{\infty} |z|^{-2-M/2} d|z|$$

which proves the lemma.

So the series $\sum v_n$ is convergent uniformly in $\operatorname{Re}\zeta > A$ for some $A > 0$ and this convergence is uniform with respect to $t, \mathbf{a}$. Then an application of the theorem on uniqueness and continuous dependence of initial conditions for linear differential equations shows that this convergence is uniform also on compacts in the right half-plane of the ζ-plane. □

Let Z_t be a family of discrete subsets of the complex plane, depending on a parameter t. We say that Z_t *depends continuously* on t if there exists a family of entire functions $f_t \neq 0$ such that Z_t is the set of zeros of f_t, and f_t depends continuously on t. Here the topology on the set of entire functions is the usual topology of uniform convergence on compact subset of the complex plane.

Consider the eigenvalue problem

$$-y'' + Qy = \lambda y, \quad y(z) \to 0, \quad z \to \infty \quad z \in L_1 \cup L_2, \tag{2.9}$$

where Q is a polynomial in z of the form (2.2), and L_1, L_2 are two rays from the origin in the complex plane. We say that a ray L is *admissible* if conditions a), b) and c) in the beginning of this section are satisfied. The notion of admissibility depends on the parameter region K participating in conditions a), b) and c).

Theorem 2.3. *If both rays L_1 and L_2 are admissible then the spectrum of problem (2.9) is continuous for* $(t, \mathbf{a}) \in [0, 1] \times K$.

Proof. Let y_1 and y_2 be the Sibuya solutions corresponding to the rays L_1 and L_2. Then their Wronski determinant $W = y_1' y_2 - y_1 y_2'$, where the primes indicate differentiation with respect to z, is the spectral determinant of the problem (2.9), and W depends continuously on $(\mathbf{a}, \lambda)$ in view of Theorem 2.1. □

The limit problem (2.9) for $t = 0$ may have no eigenvalues, this is the case when the rays L_1 and L_2 belong to adjacent sectors of $Q(z, 0, \mathbf{a})$. In this case, Theorem 2.3 says that the eigenvalues escape to infinity as $t \to 0$.

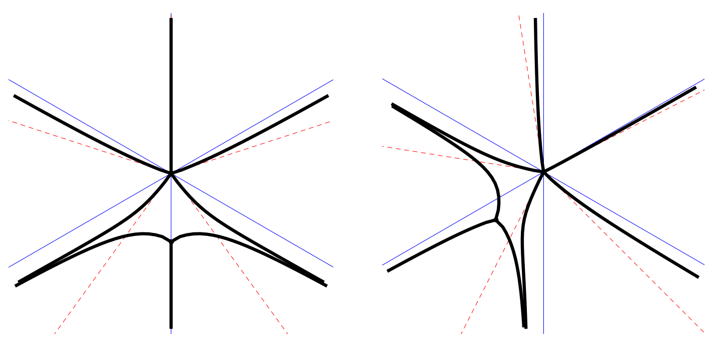
Theorem 2.4. *Consider the problem (2.9), and suppose that the rays* L_1 *and* L_2 *are admissible,* $\lambda(t, \mathbf{a})$ *is an eigenvalue that depends continuously on* $(t, \mathbf{a})$ *and has a finite limit* $\lambda(0, \mathbf{a})$ *as* $t \to 0$*. Then there exists an eigenfunction* $y(z, t, \mathbf{a})$ *of this problem, corresponding to this eigenvalue, that depends continuously on* $(t, \mathbf{a})$*.*

Proof. Let $y_1(z, t, \mathbf{a})$ be the Sibuya solution of (2.9) with $\lambda = \lambda(t, \mathbf{a})$, corresponding to the ray L_1. It depends continuously on $(t, \mathbf{a})$ by Theorem 2.4. Let $y_2(z, t, \mathbf{a})$ be the Sibuya solution of the same equation corresponding to L_2. The assumption that $\lambda(z, t, \mathbf{a}))$ is an eigenvalue implies that y_1 and y_2 are proportional. This implies that y_1 tends to zero as $z \to \infty$ on L_2, so y_1 satisfies both boundary conditions. Thus y_t is an eigenfunction that depends continuously on t. □

3. Admissible rays

In this section we give a criterion for a ray to be admissible (see the definition before Theorem 2.3). We reduce the problem to the case of a binomial $Q(z) = tz^d + cz^m,\ t \neq 0,\ c \neq 0,\ 0 < m < d$.

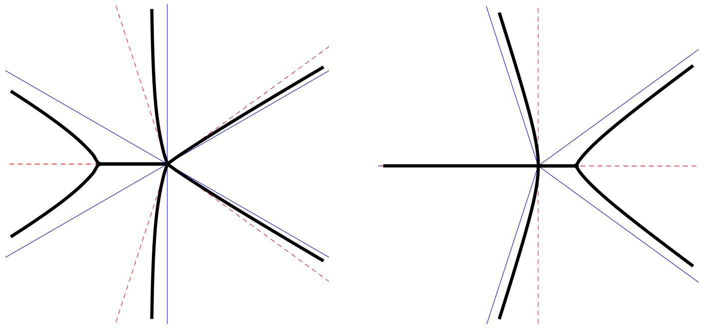
We begin with recalling terminology. Let $Q(z)$ be a polynomial, $z \in \mathbf{C}$. A *vertical line* of $Q(z)\,dz^2$ is an integral curve of the direction field $Q(z)\,dz^2 < 0$. A *Stokes line* of Q is a vertical line with one or both ends in the set of *turning points* $\{z : Q(z) = 0\}$. The *Stokes complex* of Q is the union of the Stokes lines and turning points. Examples of Stokes complexes are shown in Figs. 1-3.



(a) $z^4 + iz^3$ (b) $z^4 + e^{\pi i/4}z^3$

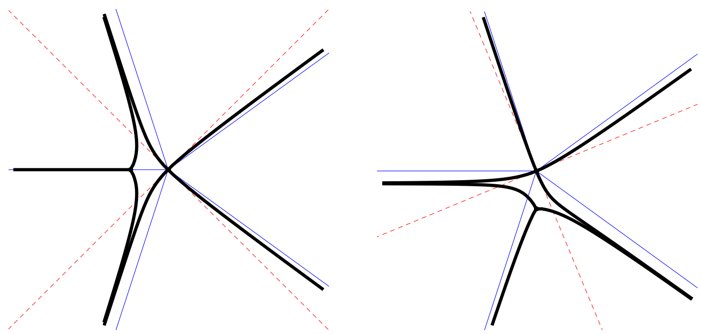
Figure 1. Stokes complexes of $z^4 + iz^3$ and $z^4 + e^{\pi i/4}z^3$.

A *horizontal line* of Q is a vertical line of $-Q$. Vertical and horizontal lines intersect orthogonally. An *anti-Stokes line* of Q is a Stokes line of $-Q$.



(a) $z^4 + z^3$ (b) $z^3 - z^2$

FIGURE 2. Stokes complexes of $z^4 + z^3$ and $z^3 - z^2$.



(a) $z^3 + z^2$ (b) $z^3 + iz^2$

FIGURE 3. Stokes complexes of $z^3 + z^2$ and $z^3 + iz^2$.

Every Stokes line has one end at a turning point and the other end either at a different turning point or at infinity. If Q has a zero at z_0 of multiplicity m then there are $m+2$ Stokes lines with the endpoint at z_0; they partition a neighborhood of z_0 into sectors of equal opening $2\pi/(m+2)$. The $m+2$ anti-Stokes lines having one end at z_0 bisect these sectors.

Let $L(\alpha) = \{z \in \mathbf{C} \setminus \{0\} : \arg z = \alpha\}$ and, for $0 < \beta - \alpha < 2\pi$, $S(\alpha, \beta) = \{z \in \mathbf{C} \setminus \{0\} : \alpha < \arg z < \beta\}$. For $R \geq 0$, let $D(R) = \{z \in \mathbf{C} : |z| \leq R\}$. For a ray L or a sector S, define $L_R = L \setminus D(R)$, $S_R = S \setminus D(R)$. For a set $S \subset \mathbf{C}$, $\overline{S}$ is its closure in $\mathbf{C}$ and $\partial S = \overline{S} \setminus S$.

Definition 3.1. Let $Q = P_d + P_m$ be a binomial, where $P_d(z) = tz^d$ and $P_m(z) = cz^m$ are two non-zero monomials, $0 < m < d$. Let $\mathcal{S}(Q)$ be the partition of $\mathbf{C} \setminus \{0\}$ into open sectors and rays defined by the Stokes lines of the two monomials P_d and P_m. Let $\mathcal{R}(Q)$ be the refinement of

$\mathcal{S}(Q)$ defined by the rays from the origin through the non-zero turning points of Q.

A ray $L(\theta)$ is called *good* for Q if it is not one of the rays of $\mathcal{R}(Q)$ and is not tangent to any vertical line of Q. The last condition is equivalent to $L(\theta) \cap Z = \emptyset$ where $Z = \{z : z^2Q(z) \le 0\}$. A sector S of $\mathcal{R}(Q)$ is *good* for Q if each ray $L(\theta) \subset S$ is good. This is equivalent to $S \cap Z = \emptyset$.

Theorem 3.2. *Let $Q = P_d + P_m$ be as in Definition 3.1. Then any sector of $\mathcal{R}(Q)$ containing an anti-Stokes line of P_d is good for Q, and any good ray belongs to one of such sectors.*

Proof. Let $\mathbf{R}_+$ and $\mathbf{R}_-$ be the positive and negative real rays (not including 0). Definition 3.1 implies that a ray $L = L(\theta)$ is good if and only if the cone $C_L = \{\alpha z^2 P_d(z) + \beta z^2 P_m(z),\ z \in L,\ \alpha \in \mathbf{R}_+,\ \beta \in \mathbf{R}_+\}$ does not contain $\mathbf{R}_-$.

An anti-Stokes line of P_d is a good ray unless it is also a Stokes line of P_m, because $z^2P_d(z)$ is real positive on it, hence $z^2P_m(z)$ must be real negative to make the sum real negative.

Suppose that L is an anti-Stokes line of P_d, and that $z^2P_m(z)$ is either real positive or belongs to the upper half-plane for $z \in L$. Then either $C_L = \mathbf{R}_+$ or C_L belongs to the upper half-plane. When L is rotated counterclockwise, the arguments of the two monomials z^2P_d and z^2P_m restricted to L are increasing. Hence C_L remains in the upper half-plane until at least one of the monomials becomes real negative on L, i.e., until L becomes a Stokes line of either P_d or P_m, or both. When L is rotated clockwise, the arguments of the two monomials restricted to L are decreasing. Since the argument of P_d decreases faster than the argument of P_m, the cone C_L does not contain negative real numbers until either z^2P_d on L becomes real negative or $\arg P_m - \arg P_d$ passes π, i.e., until L either becomes a Stokes line of P_d or passes a non-zero turning point of Q.

The case when $P_m(z)$ belongs to the lower half-plane on an anti-Stokes line of P_d is done similarly.

Conversely, let L be a ray which is not one of the rays of $\mathcal{R}(Q)$ and such that C_L does not contain negative real numbers. Then either L itself is an anti-Stokes line of P_d, or it can be rotated to the closest anti-Stokes line of P_d preserving this property.

For example, if $z^2P_d(z)$ belongs to the upper half-plane for $z \in L$, then $\arg(z^2P_d(z)) - \pi < \arg(z^2P_m(z)) < \pi$ for $z \in L$. Otherwise, either L would be a Stokes line of P_m (if $\arg(z^2P_m(z)) = \pi$), or it would contain a turning point of Q (if $\arg(z^2P_d(z)) - \pi = \arg(z^2P_m(z))$), or C_L would contain $\mathbf{R}_-$. When L is rotated clockwise, since $\arg P_d$

decreases faster than $\arg P_m$, P_d would become real positive on L before either $\arg P_m - \arg P_d$ becomes π or $z^2 P_m$ becomes real negative.

The case when $z^2 P_d(z)$ belongs to the lower half-plane on L is done similarly, rotating L counterclockwise. □

Corollary 3.3. *Let $S = S(\alpha, \beta)$ be a Stokes sector of P_d such that the anti-Stokes line $L \subset S$ of P_d does not contain a turning point of Q. Then S contains a good subsector.*

Proof. Since L does not contain a turning point of Q, it cannot be a Stokes line of P_m. Hence L belongs to a sector of $\mathcal{R}(Q)$, which is good by Theorem 3.2. □

Theorem 3.4. *Let $Q = P_d + P_m$ be as in Definition 3.1. Let $H = \{(x, y) \in \mathbf{R}^2 : |x| < \pi,\ |y| < \pi,\ |y - x| < \pi\}$ be a hexagon in $\mathbf{R}^2$. Then $\mathbf{R}_+$ is a good ray for Q if and only if* $(\arg t,\ \arg c) \in H$*. Here the values of* $\arg t$ *and* $\arg c$ *are taken in* $(-\pi, \pi]$*.*

A ray $L(\theta)$ is a good ray for Q if and only if $(\arg t,\ \arg c)$ *belongs to H translated by $(2\pi k - (d + 2)\theta,\ 2\pi l - (m + 2)\theta)$ for some integers k and l.*

Proof. For $t > 0$ and $c > 0$, $\mathbf{R}_+$ is an anti-Stokes line of both P_d and P_m, hence it is a good ray. It is not a Stokes line of either P_d or P_m when $|\arg t| < \pi$ and $|\arg c| < \pi$. It does not contain a turning point if $|\arg t - \arg c| < \pi$.

For $|\arg t| < \pi$, the anti-Stokes line L of P_d closest to $\mathbf{R}_+$ has the argument $-\arg t/(d + 2)$. It is a Stokes line of P_m if $\arg c - (m + 2) \arg t/(d + 2) = \pi(2k + 1)$ for an integer k. Since $0 < m < d$, the lines $y - (m + 2)x/(d + 2) = \pi(2k + 1)$ do not intersect H. Hence L does not contain a turning point of Q when $(\arg t,\ \arg c) \in H$. This implies that the closure of the sector S bounded by $\mathbf{R}_+$ and L does not contain Stokes lines of either P_d or P_m, and does not contain non-zero turning points of Q, for all (t, c) such that $(\arg t,\ \arg c) \in H$. Due to Theorem 3.2, $\mathbf{R}_+$ is a good ray for Q with these values of t and c.

If one of the three inequalities defining H becomes an equality, $\mathbf{R}_+$ becomes either a Stokes line of one of the two monomials of Q, or contains a turning point of Q.

When $(\arg t,\ \arg c)$ crosses one of the two segments $|\arg t| < \pi$, $|\arg c| < \pi$, $|\arg t - \arg c| = \pi$ of the boundary of H, a turning point of Q crosses $\mathbf{R}_+$ and remains inside the sector S for all values (t, c) such that $|\arg t| < \pi,\ |\arg c| < \pi,\ |\arg c - (m + 2) \arg t/(d + 2)| < \pi$. Due to Theorem 3.2, $\mathbf{R}_+$ is not a good ray for Q with these values of t and c.

This implies that $\mathbf{R}_+$ is not a good ray for Q when $(\arg t,\ \arg c) \notin H$.

The statement for $L(\theta)$ is reduced to the statement for $\mathbf{R}_+$ by the change of variable $z = ue^{i\theta}$ in the quadratic differential $Q(z)\,dz^2$. □

Example 3.5. Let us investigate when the two rays of the real axis are good for a binomial $Q(z) = tz^d + cz^m$. According to Theorem 3.4, $\mathbf{R}_+$ is a good ray when $(\arg t, \arg c) \in H$ (green hexagon in Fig. 4a).

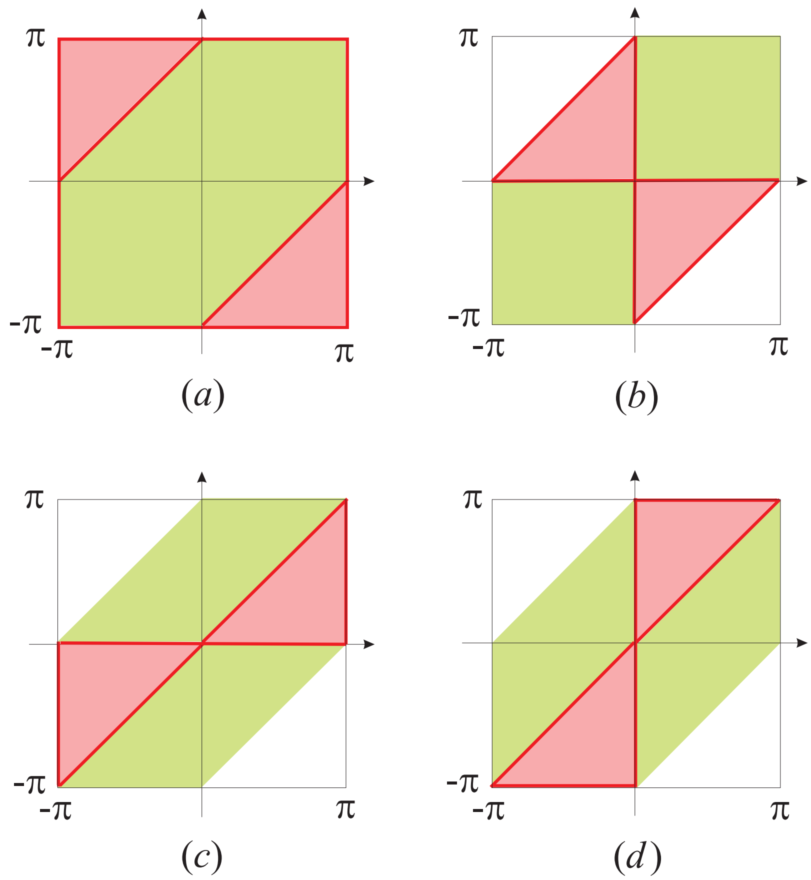


FIGURE 4. The values of $(\arg t, \arg c)$ in Example 3.5 where both $\mathbf{R}_+$ and $\mathbf{R}_-$ are good (green) and where $\mathbf{R}_-$ is not good (red). (a) d and m even; (b) d and m odd; (c) d even, m odd; (d) d odd, m even.

For $\mathbf{R}_-$ ($\theta = \pi$ in Theorem 3.4) there are 4 cases, depending on the parity of d and m.

a) If d and m are even, both $\mathbf{R}_+$ and $\mathbf{R}_-$ are good rays for any Q such that $(\arg t, \arg c) \in H$.

b) If d and m are odd, $\mathbf{R}_-$ is a good ray for Q when $(\arg t, \arg c)$ belongs to the complement in $(-\pi, \pi]^2$ of the two triangles (red area in

Fig. 4b) with the vertices $(0,0)$, $(0,\pi)$, $(-\pi,0)$ and $(0,0)$, $(\pi,0)$, $(0,-\pi)$, respectively. Both $\mathbf{R}_+$ and $\mathbf{R}_-$ are good rays for Q when $(\arg(t),\arg(c))$ belongs to the union of the two squares $(0,1)^2$ and $(-1,0)^2$ (green area in Fig. 4b).

c) If d is even and m is odd, $\mathbf{R}_-$ is a good ray for Q when $(\arg t,\arg c)$ belongs to the complement in $(-\pi,\pi]$ of the two triangles (red area in Fig. 4c) with the vertices $(0,0)$, $(\pi,0)$, (π,π) and $(0,0)$, $(-\pi,0)$, $(-\pi,-\pi)$, respectively. Both R_+ and R_- are good rays for Q when $(\arg(t),\arg(c))$ belongs to the union of the two open parallelograms (green area in Fig. 4c) with the generators $(\pi,0)$, $(-\pi,-\pi)$ and (π,π), $(-\pi,0)$, respectively.

d) If d is odd and m is even, $\mathbf{R}_-$ is a good ray for Q when $(\arg t,\arg c)$ belongs to the complement in $(-\pi,\pi]^2$ of the two triangles (red area in Fig. 4d) with the vertices $(0,0)$, $(0,\pi)$, (π,π) and $(0,0)$, $(0,-\pi)$, $(-\pi,-\pi)$, respectively. Both R_+ and R_- are good rays for Q when $(\arg(t),\arg(c))$ in the union of the two open parallelograms (green area in Fig. 4d) with the generators $(0,\pi)$, $(-\pi,-\pi)$ and $(0,-\pi)$, (π,π), respectively.

In particular, if t is on the positive imaginary axis, both R_+ and R_- are good rays for Q when $-\pi/2<\arg(c)<\pi$ in case (a), $0<\arg(c)<\pi$ in case (b), $-\pi/2<\arg(c)<0$ or $\pi/2<\arg(c)<\pi$ in case (c), $-\pi/2<\arg(c)<\pi/2$ in case (d).

By definition, a good ray L is not tangent to the vertical lines of Q and is not a Stokes line of either P_d or P_m. Since the angles between L and vertical lines of Q have non-zero limits at the origin and at infinity, there is a lower bound for these angles on L. This lower bound depends continuously on L, hence there is a common lower bound for these angles for all rays in a proper subsector T of a good sector S (such that $\overline{T}\setminus\{0\}\subset S$).

The good sectors in Theorem 3.2 depend continuously on the arguments of the coefficients t and c of monomials P_d and P_m, except when a good sector degenerates to a ray that is a Stokes line of P_m and an anti-Stokes line of P_d.

The lower bounds for the angles between a good ray L and vertical lines, and for the values of R, depend continuously on the arguments of t and c, except when a good sector containing L degenerates.

Now we show that a good ray for a monomial tz^d+cz^m is admissible for the potential (2.2), with

$$K=\{(a_0,\ldots,a_{m-1},c):\sup_{0\leq j\leq m-1}|a_j|\leq M\},$$

for every positive M.

Lemma 3.6. *Consider the polynomials* $Q(z) = tz^d + cz^m + q(z)$, *where* $t \geq 0$, $|q(z)| \leq M|z|^{m-1}$ *and* $|c| = 1$. *Let* S *be a sector whose closure does not contain turning points of* $tz^d + cz^m$.

For every $\epsilon > 0$ *there exists* $R > 0$ *depending on* ϵ, M, S, *such that for every* $t \geq 0$:

(i) The set $S \cap \{z : |z| \geq R\}$ *does not contain turning points of* Q, *and*

(ii) If $v(z)$ *and* $v'(z)$ *are the vertical directions of* $Q(z)dz^2$ *and* $(tz^d + cz^m)dz^2$, *respectively, then* $|\arg v(z) - \arg v'(z)| < \epsilon$ *for* $z \in S$, $|z| > R$.

Proof. We have $|tz^d + cz^m| \geq c|z|^m$ for $z \in S$, so $|q(z)/(tz^d + cz^m)| \leq c^{-1}|z|^{-1}$, and (i), (ii) hold when R is large enough. □

4. PT-symmetric potentials and linear differential equations having solutions with prescribed number of non-real zeros

Hellerstein and Rossi asked the following question [9, Problem 2.71]. Let

$$w'' + Pw = 0 \tag{4.1}$$

be a linear differential equation with polynomial coefficient P. Characterize all polynomials P such that the differential equation admits a solution with infinitely many zeros, all of them real.

This problem was investigated in [42, 29, 28, 33, 38, 22]. Recently K. Shin [39] announced a description of polynomials P of degree 3 or 4 such that equation (4.1) has a solution with infinitely many zeros, *all but finitely many of them* real. It turns out that equations (4.1) with this property are equivalent to (1.4) or (1.7) of the Introduction by an affine change of the independent variable.

Here we use the methods of [21, 19] to parametrize polynomials P of degrees 3 and 4 such that equation (4.1) has a solution with *prescribed* number of non-real zeros.

We begin with degree 3.

Theorem 4.1. *For each integer* $n \geq 0$ *there exists a simple curve* Γ_n *in the plane* $\mathbf{R}^2$ *which is the image of a proper analytic embedding of the real line and which has the following properties.*

For every $(a, \lambda) \in \Gamma_n$ *the equation*

$$-w'' + (z^3 - az + \lambda)w = 0 \tag{4.2}$$

has a solution w *with* $2n$ *non-real zeros. Real zeros belong to a ray* $(-\infty, x_0)$ *and there are infinitely many of them. This solution satisfies* $\lim_{t\to\pm\infty} w(it) = 0$.

The union $\cup_{n=0}^{\infty}\Gamma_n$ *coincides with the real part of the spectral locus of (1.4).*

The projection $(a,\lambda)\mapsto a$,

$$\Gamma_n \cap \{(a,\lambda) : a \geq 0\} \to \{a : a \geq 0\}$$

is a 2*-to-*1 *covering map. The curves* Γ_n *are disjoint, and for* $a \geq 0$ *and* $n \geq 0$*, if* $(a,\lambda) \in \Gamma_n$ *and* $(a,\lambda') \in \Gamma_{n+1}$ *then* $\lambda < \lambda'$.

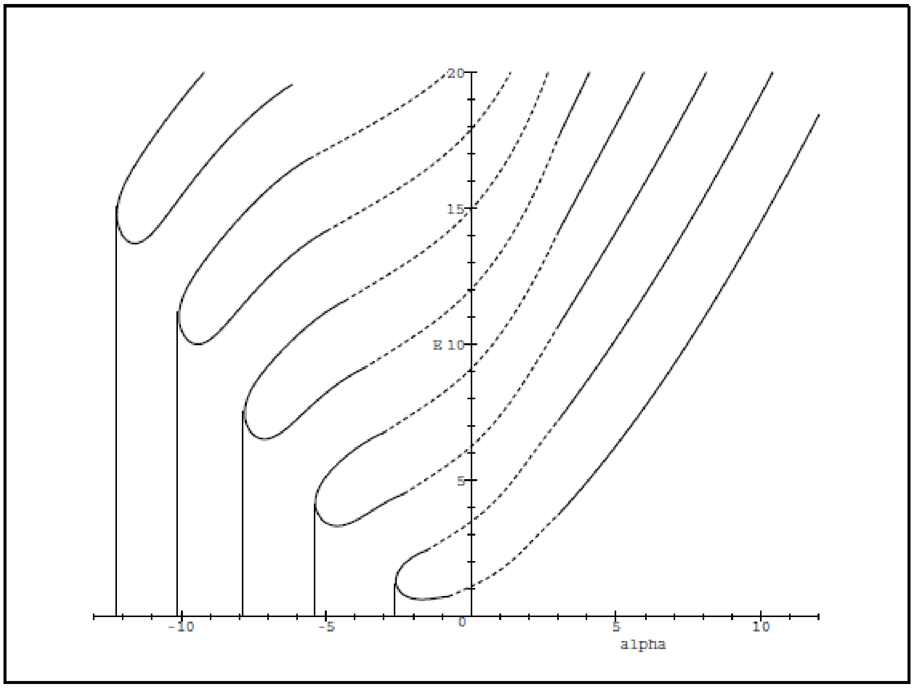


FIGURE 5. Curves $\Gamma_n,\ n = 0,\ldots,4$ in the (a,λ) plane (Trinh, 2002).

Equation (4.2) is equivalent to the PT-symmetric equation (1.4) in the Introduction by the change of the independent variable $z \mapsto iz$. Computer experiments strongly suggest that the projection $(a,\lambda)\mapsto a$ is 2-to-1 on the whole curve Γ_n except one critical point of this projection, and that the whole curve Γ_{n+1} lies above Γ_n.

Fig. 5, taken from Trinh's thesis [43] (see also [14]), shows a computer generated picture of the curves Γ_n.

As a corollary from Theorem 4.1 we obtain that every eigenvalue $\lambda_n(a),\ a \geq 0$ of (1.4), when analytically continued to the left along the a-axis, encounters a singularity for some $a < 0$. According to Theorem 2 of [19] this singularity is an algebraic ramification point.

Proof. Consider the Stokes sectors of equation (4.2). We enumerate them counter-clockwise as $S_0, \dots S_4$ where S_0 is bisected by the positive real axis. Consider the set G of all real meromorphic functions f whose Schwarzian derivatives are real polynomials of the form $-2z^3+a_2z^2+a_0$, and whose asymptotic values in the sectors $S_0, \dots, S_4$ are $\infty, 0, b, \overline{b}, 0$, respectively, where $b = e^{i\beta}, \beta \in (0, \pi)$. Such functions are described by certain cell decompositions of the plane [19]. By a cell decomposition we understand a representation of a space X as a union of disjoint cells. This union is locally finite, and the boundary of each cell consists of cells of smaller dimension. The 0-cells are points, vertices of the decomposition. The 1-cells are embedded open intervals, the edges, and the 2-cells are embedded open discs, faces of a decomposition. Two cell decompositions of a space X are called equivalent if they correspond to each other via an orientation-preserving homeomorphism of X.

To describe functions of the set G, we begin with the cell decomposition Φ of the Riemann sphere shown in Fig. 6.

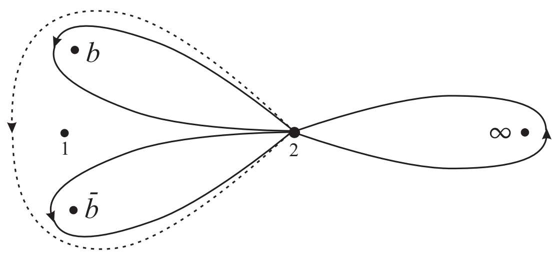


FIGURE 6. Cell decomposition Φ of the image sphere (solid lines).

It consists of one vertex at 2 and three edges which are simple disjoint loops around $b, \overline{b}$ and ∞, so that the loop around ∞ is symmetric with respect to complex conjugation while the loops about b and $\overline{b}$ are interchanged by the complex conjugation. The point 0 is outside the Fig. 6. The dotted line is not discussed here; it is needed for the future. Also for the future use, we assume that the loop around ∞ passes through the point -2, and that this loop is symmetric with respect to the reflection $z \mapsto -\overline{z}$.

So our cell decomposition has one vertex, three edges and four faces. The faces are labeled by points $b, \overline{b}, \infty$ and 0 which are inside the faces. (So three faces are bounded by single edge each, while one face (labeled with 0) is bounded by three edges).

Suppose now that we have a local homeomorphism $g : \mathbf{C} \to \overline{\mathbf{C}}$ such that the restriction

$$g : \mathbf{C} \backslash g^{-1}(A) \to \overline{\mathbf{C}} \backslash A, \tag{4.3}$$

where $A = \{b, \overline{b}, \infty, 0\}$, is a covering map, and $\Psi = g^{-1}(\Phi)$. Then we preimage $\Psi = g^{-1}(\Phi)$ will be a cell decomposition of the plane $\mathbf{C}$. Now suppose that a cell decomposition Ψ of the plane is given in advance, and suppose that its local structure is the same as that of Φ. This means that the faces of Ψ are labeled by elements of the set A, and that a neighborhood of each vertex of Ψ can be mapped onto a neighborhood of the vertex of Φ by an orientation-preserving homeomorphism, respecting the labels of the faces. Then there exists a local homeomorphism $g : \mathbf{C} \to \overline{\mathbf{C}}$ such that (4.3) is a covering map. We use the following cell decomposition to construct g:

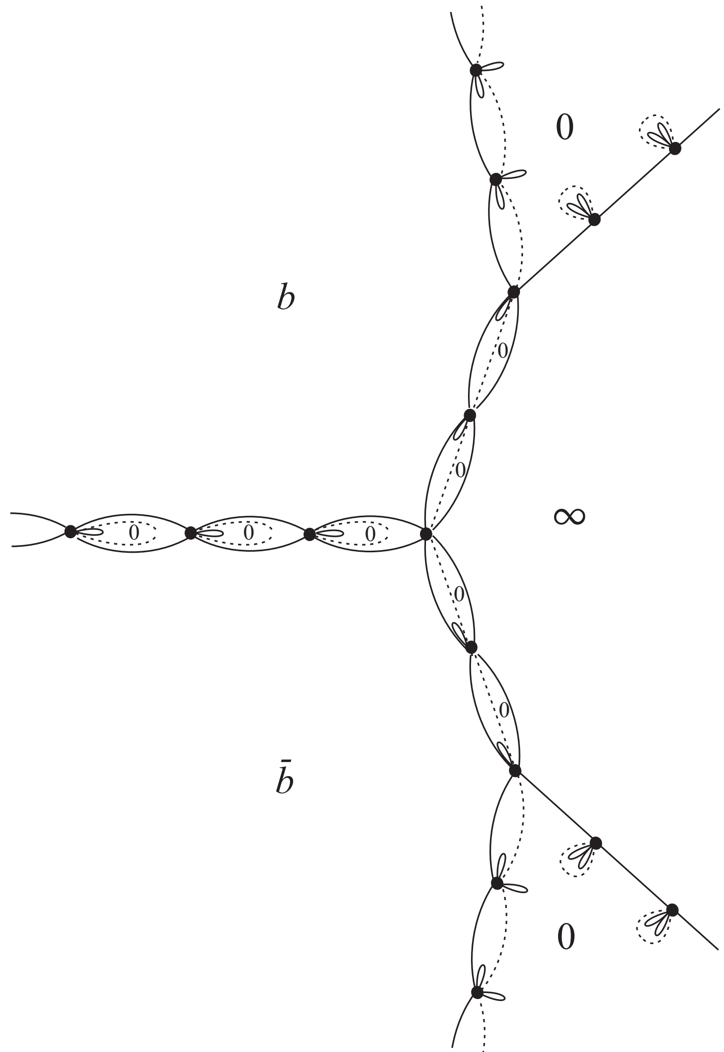


FIGURE 7. Cell decomposition Ψ_n for $n = 2$ (solid lines).

The five "ends" extend to infinity periodically. This cell decomposition depends on one integer parameter $n \geq 0$ which is the number of 0-labeled faces between the neighboring "ramification points". Only some face labels are shown but the reader can easily recover all other labels from the condition that in a neighborhood of each vertex Ψ_n is similar to a neighborhood of the vertex of Φ. The dotted lines are not a part of our cell decomposition; they are preimages of the dotted line in Fig. 19, and are added for a future need. Ψ_n is symmetric with respect to the real line, this permits to make our local homeomorphism g symmetric, that is $g(\overline{z}) = \overline{g(z)}$. This construction defines the map g up to pre-composition with a symmetric homeomorphism $\phi : \mathbf{C} \to \mathbf{C}$ of the domain of g. A fundamental result of R. Nevanlinna ensures that this homeomorphism ϕ can be chosen in such a way that $f = g \circ \phi$ is a meromorphic function which is real in the sense that $f(\overline{z}) = \overline{f(z)}$. We refer to [19] for the discussion of this construction in our current context; in fact [19] contains a simple alternative proof of Nevanlinna's theorem. Nevanlinna's original proof is explained in modern language in [18]; the original paper of Nevanlinna is [36].

The meromorphic function f is defined by the cell decomposition Ψ_n and parameter b up to pre-composition with a real affine map $cz + d$. Furthermore, the Nevanlinna theory says that the Schwarzian derivative of f is a polynomial of degree exactly 3 (the number of unbounded faces of Ψ_n minus 2). We pre-compose f with a real affine map to normalize this polynomial to have leading coefficient -2 and zero coefficient at z^2. Thus

$$\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 = -2(z^3 - az + \lambda). \tag{4.4}$$

As f is real, a and λ are also real. Now f is uniquely defined by the properties that it satisfies a differential equation (4.4), has asymptotic values $\infty, 0, b, \overline{b}, 0$ in the sectors $S_0, \ldots, S_4$, respectively, and that $f^{-1}(\Phi)$ equivalent to Ψ_n (Fig. 7 for $n = 2$) by an orientation-preserving homeomorphism of the plane commuting with the reflection $z \mapsto \overline{z}$. The statement on asymptotic values implies that

$$f(it) \to 0, \quad t \to \pm\infty.$$

Furthermore, f depends analytically on b, when b is in the upper half-plane, and thus we obtain a real analytic map $b \mapsto (a, \lambda)$. This map is evidently invariant with respect to transformations $b \mapsto tb,\ t \in \mathbf{R} \backslash \{0\}$, this is because the Schwarzian derivative in the right hand side of (4.4) does not change when f is replaced by tf.

Thus for every n, we have a one-parametric family $G_n \subset G$ of meromorphic functions, parametrized by $\beta \in (0, \pi)$, $b = e^{i\beta}$. Taking the Schwarzian derivative we obtain a map $F_n : (0, \pi) \to \mathbf{R}^2$, $\beta \mapsto (a, \lambda)$. This map is known to be a proper real analytic immersion [2]. It is easy to see that it is injective: two solutions of the same Schwarz equation may differ only by post-composition with a fractional-linear map, and this fractional-linear map must be identity by our normalization of asymptotic values.

For the same reasons the images of F_n are disjoint: for different n, our functions have (topologically) different line complexes. The images of F_n are our curves Γ_n.

Now we prove that the union of Γ_n coincides with the real part of the spectral locus of (1.7).

Our functions $f \in G$ can be written in the form $f = w/w_1$ where w and w_1 are two linearly independent solutions of equation (4.2) with some real a and λ. We can choose w and w_1 to be real entire functions. Condition that $f(it) \to 0$ as $t \to \pm\infty$ implies that $w(it) \to 0$ for $t \to \pm\infty$ so w is an eigenfunction of the spectral problem

$$-w'' + (z^3 - az + \lambda)w = 0, \quad w(\pm i\infty) = 0, \tag{4.5}$$

which is equivalent to (1.4) by the change of the independent variable $z \mapsto iz$.

So our curves Γ_n belong to the real part of the spectral locus of (4.5) or (1.4).

Now, let λ be a real eigenvalue of the problem (4.5), w a corresponding eigenfunction. Choose a point x_0 on the real line such that $w(x_0) \neq 0$ and normalize w so that $w(x_0) = 1$. Then $w^*(z) = \overline{w(\overline{z})}$ is an eigenfunction with the same eigenvalue, so $w^* = cw$ for some constant $c \neq 0$. Substituting x_0 gives that $c = 1$. So w is real.

Let w_1 be a solution of the same equation as w but satisfying $w(x) \to 0$, $x \to +\infty$. We normalize w_1 so that w_1 is real in the same way as we normalized w. Then $f = w/w_1$ is a real meromorphic function whose Schwarzian derivative is a cubic polynomial with top coefficient -2, and the asymptotic values in S_j are $\infty, 0, b, \overline{b}, 0$. We can change the normalization of w_1 multiplying it by any real non-zero constant. In this way we achieve that $b = e^{i\beta}$ for some $\beta \in (0, \pi)$. So f belongs to the class G.

Lemma 4.2. $G = \cup_{n=0}^{\infty} G_n$.

Proof. Let $f \in G$. Consider the cell decomposition $X = f^{-1}(\Phi)$. We have to prove that $X = \Psi_n$ for some $n \geq 0$. To do this, we follow [19]. We first remove all loops from X, and then replace each multiple edge by a single edge, and denote the resulting cell decomposition by Y. Notice that the cyclic order $(\infty, b, \overline{b})$ in Fig. 6 is consistent with the cyclic order $(\infty, 0, b, \overline{b}, 0)$ of the Stokes sectors in the z-plane. By [19, Proposition 6], this implies that the 1-skeleton of Y is a tree. This infinite tree is properly embedded in the plane, has 5 faces, is symmetric with respect to the real line, and has two faces labeled with 0 which are interchanged by the symmetry. Moreover, the faces of Y are in one-to-one correspondence with the Stokes sectors, and the face corresponding to S_0 is bisected by the positive ray. One can easily classify all trees with these properties. They depend of one integer parameter $n \geq 0$ which is the distance between the ramification point in the upper half-plane and the ramification point on the real axis. Now we refer to [19, Proposition 7] that the tree Y uniquely defines the cell decomposition X. This shows that $X = \Psi_n$ for some $n \geq 0$.

Meromorphic function f is defined by the cell decomposition X and the parameter b up to an affine change of the independent variable. Normalizing it as in (4.4) gives $f \in G_n$. □

We conclude that the union of our curves Γ_n in the right half-plane $a \geq 0$ coincides with the real part of the spectral locus of (4.5).

Now we study the shape of the curves Γ_n. The boundary value problem (4.5) was considered by Shin [37], Delabaere and Trinh [43, 14]. The spectrum of this problem is discrete, simple and infinite. It is known [37] that for $a \geq 0$ all eigenvalues of this problem are real and positive. It follows from this result that there are real analytic curves $\lambda = \gamma_k(a),\ k = 0, 1, 2, \ldots$, such that for each k, $\gamma_k(a)$ is an eigenvalue of the problem (4.5), and $\gamma_k < \gamma_{k+1},\ k = 0, 1, 2, \ldots$. So the part of the real spectral locus in $\{(a, \lambda) : a \geq 0\}$ is the union of the graphs of γ_k.

Next we prove that the intersection of Γ_n with the half-plane $a \geq 0$ consists of γ_{2n} and γ_{2n+1}. For this purpose we study what happens to eigenvalues and eigenfunctions of the problem (4.5) as $a \to +\infty$.

A different approach to the asymptotics as $a \to \infty$ is used in [27]. We could use their Corollary 2.16 here instead of referring to Sections 2,3.

We substitute $cz + d$ in (4.5) and put $y(z) = w(cz + d)$, where

$$d = (a/3)^{1/2} > 0, \quad c = (3d)^{-1/4} > 0.$$

The result is

$$-y'' + (c^5 z^3 + z^2 + \mu)y = 0, \tag{4.6}$$

where $\mu = c^2(\lambda + d^3 - ad)$. Choosing the positive and negative imaginary rays as our normalization rays L_1 and L_2, we see that the normalization rays are admissible in the sense of Theorem 2.3. The Stokes complex of the binomial potential corresponding to (4.6) is shown in Fig. 3(a). According to Theorem 2.3, the spectrum of the problem (4.6) converges to the spectrum of the limit problem

$$-y'' + z^2 y = -\mu y, \quad y(\pm i\infty) = 0. \tag{4.7}$$

This limit problem is equivalent to the self-adjoint problem

$$-u'' + z^2 u = \mu u, \quad u(\pm\infty) = 0$$

by the change of the variable $u(z) = y(iz)$. Convergence of the spectrum implies convergence of eigenfunctions uniform on compact subsets of the plane by Theorem 2.4. As a varies from 0 to ∞, we can choose an eigenvalue $\lambda(a)$ which varies continuously, and the corresponding eigenfunction that varies continuously, and tends to an eigenfunction of (4.6). In the process of continuous change the number of non-real zeros of the eigenfunction cannot change because eigenfunctions cannot have multiple zeros. The conclusion of the theorem will now follow from the known properties of zeros of eigenfunctions of Hermitian boundary value problems, once we establish the following

Lemma 4.3. *As $t = c^5 \to 0$ in (4.6) the non-real zeros of an eigenfunction cannot escape to infinity.*

Notice that the real zeros of the eigenfunction do escape to infinity, as the limit eigenfunction has at most one real zero.

Proof. Let w_t be the eigenfunction constructed in Theorem 2.4 which depends continuously on t. Let w_t^* be the Sibuya solution corresponding to the positive ray. Then $f_t = w_t/w_t^*$ is a real meromorphic solution of the Schwarz equation and has asymptotic values $\infty, 0, b_t, \overline{b_t}, 0$ in the sectors S_j. As $f_t \to f_0$, and the Schwarzian of f_0 is of degree 2, we conclude that $b = e^{i\beta}$ converges to the real axis, and the Riemann surface of f_t^{-1} must converge in the sense of Caratheodory [10, 45], to a Riemann surface with 4 logarithmic branch points which can lie only over $0, \infty, b_0$, where $b_0 \in \{\pm 1\}$. To construct the cell decomposition corresponding to this limit Riemann surface, we consider two cases.

Case 1. $b_0 = 1$. To describe the limit function, we must replace in the original cell decomposition Fig. 6 two loops corresponding to b, $\overline{b}$ with a single loop around both of these points. This loop is shown by the dotted line in Fig. 6 and its preimage is shown by the dotted lines in Fig. 7. The original loops that separate b- and $\overline{b}$- labeled faces from the face labeled 0 must be removed. Performing this operation on the cell decomposition Ψ_n we see that the 1-skeleton breaks into infinitely many pieces. But there is only one piece that has four unbounded faces and thus can correspond to a meromorphic function whose Schwarzian derivative is a polynomial of degree 2. This limit decomposition is shown in Fig. 8.

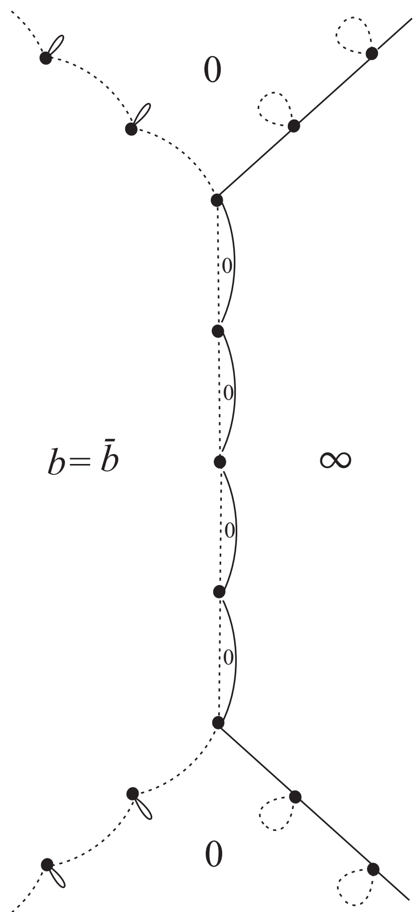


FIGURE 8. Limit cell decomposition with $n = 2$ (solid and dotted lines).

This time both solid and dotted lines represent the edges of this decomposition. We see that the number of non-real zeros in the limit is the same as it was before the limit.

Case 2 $b_0 = -1$. To analyze this case, we replace the cell decomposition on Fig. 6 by the one in Fig. 9

Now we want to find the preimage of the cell decomposition in Fig. 9 under the same function f. There are several ways to find this preimage. Let us choose for convenience $b = i$, and express the loops in

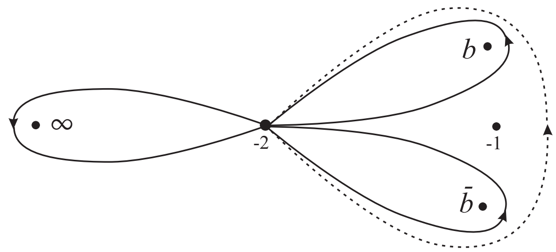


FIGURE 9. Another cell decomposition of the sphere (solid lines).

Fig. 9, in terms of the loops in Fig. 6, as elements of the fundamental group of $\mathbf{C}\backslash\{0, i, -i\}$. We denote the loops around b,$\overline{b}$ in Fig. 6 by $\gamma_b, \gamma_{\overline{b}}$, and let α, β be the upper and lower halves of the loop around ∞, so that $\gamma_\infty = \alpha\beta$ (α followed by β). Let γ'_b and $\gamma'_{\overline{b}}$ be the loops in Fig. 9. Then we have $\gamma_\infty = \gamma'_\infty = \beta\alpha$ (α followed by β), $\gamma'_b = \beta\gamma_b\beta^{-1}$, $\gamma'_{\overline{b}} = \alpha^{-1}\gamma_{\overline{b}}\alpha$. See Fig. 10.

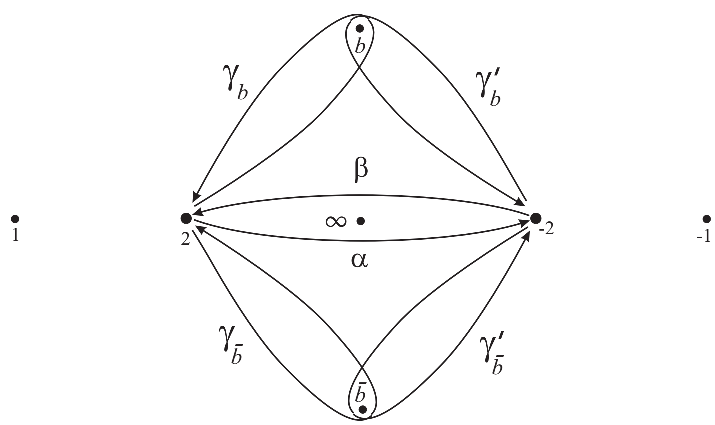


FIGURE 10. Fig. 6 and Fig. 9 together.

These relations permit us to draw the preimages of the loops $\gamma'_\infty, \gamma'_b, \gamma'_{\overline{b}}$ in the z-plane. The resulting picture is shown in Fig. 11.

When $b \to -1$, we have a degeneration as before. The corresponding cell decomposition is obtained by replacing preimages of the loops γ'_b and $\gamma'_{\overline{b}}$ by the preimages of the dotted line. The resulting cell decomposition is shown in Fig. 12.

We see that the limit function still has $2n$ non-real zeros, and one real zero. This completes the proof of the lemma, and of Theorem 4.1. □

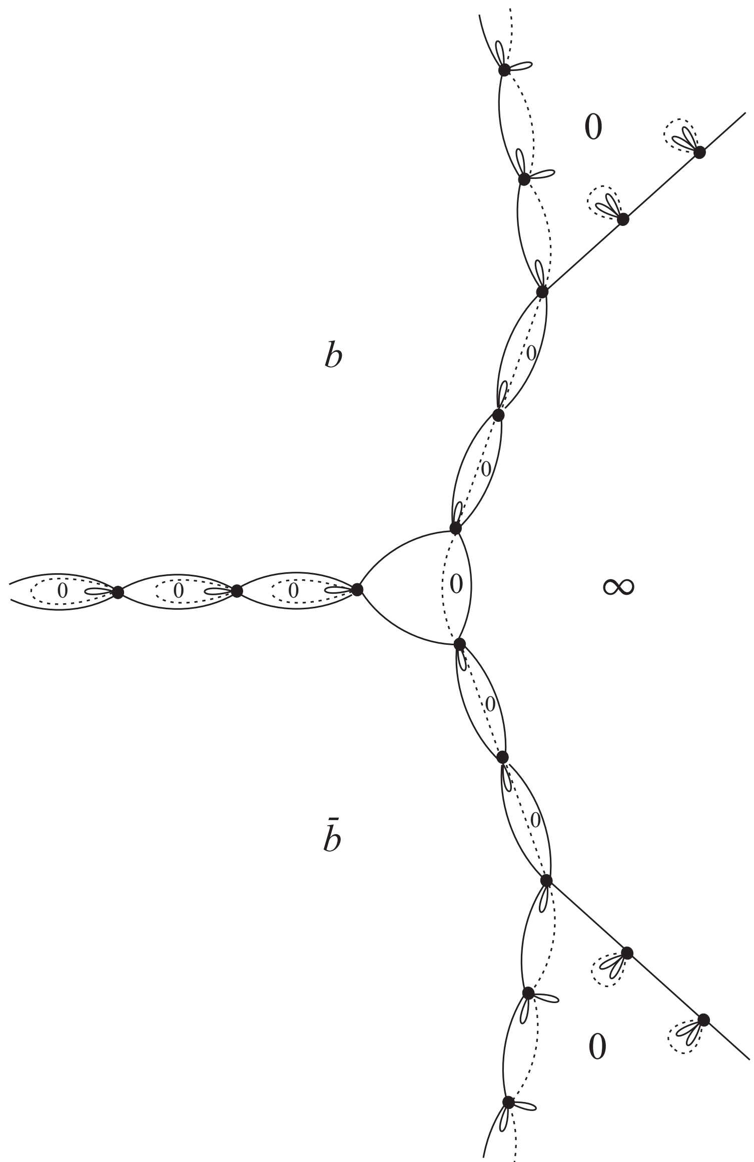


FIGURE 11. Preimage of the cell decomposition Fig. 9 (solid lines).

This proof shows that $\beta \to 0$ corresponds to the lower branch of Γ_n while $\beta \to \pi$ corresponds to the upper branch.

$\square$

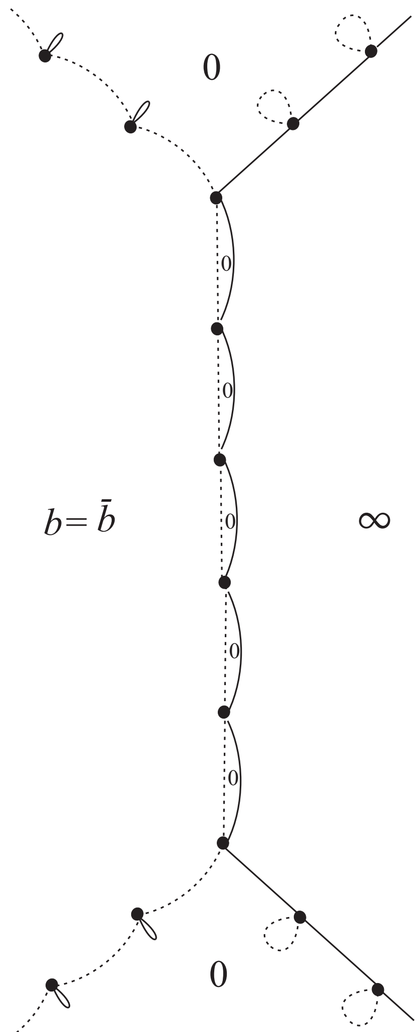


FIGURE 12. The limit cell decomposition of Fig. 10 as $b \to -1$ (solid and dotted lines).

Theorem 4.4. *Let P be a polynomial of degree 3 such that equation (4.1) has a solution with $2n$ non-real zeros. Then (4.1) can be transformed to an equation of Theorem 4.1 with $(a, \lambda) \in \Gamma_n$ by a real affine change of the independent variable.*

Proof. By the results of Gundersen [28, 29], all solutions have infinitely many zeros, and the coefficients of P are real. By a real affine change of the variable we achieve that $P(z) = -z^3 + az - \lambda$. As almost all zeros are real, our solution must tend to zero in both directions of the imaginary axis. So λ is an eigenvalue of the problem (4.5). Let w be a real eigenfunction and w_1 a real solution of our equation that is linearly independent of w. Then the ratio $f = w/w_1$ is a meromorphic function which is a local homeomorphism, and has asymptotic values $\infty, 0, b, \overline{b}, 0$ in $S_0, \ldots, S_4$, respectively. After a real affine change of the independent variable, this function will belong to the class G defined in the proof of Theorem 4.1. □

Now we state analogous results for quartic oscillators. There are two different real two-parametric families in which solutions with finitely many non-real zeros can occur [39].

$$(4.8) \qquad -w'' + (-z^4 + az^2 + cz + \lambda)w = 0, \quad w(\pm i\infty) = 0.$$

studied in [3, 13], and

$$(4.9) \quad -w'' + (z^4 - 2az^2 + 2mz + \lambda)w = 0, \quad w(te^{i\theta}) \to 0, \quad \theta = \pm\pi/3$$

studied in [4]. Here $m \geq 1$ is an integer. Problem 4.9 is quasi-exactly solvable, which means that there are m eigenfunctions of the form $p(z)\exp(z^3/3 - az)$, where p is a polynomial of degree $m - 1$. The families (4.8) and (4.9) are equivalent to the PT symmetric families (1.7) and (1.8-1.9) of the Introduction via the change of the independent variable $z \mapsto iz$.

Theorem 4.5. *The real part of the spectral locus of (4.8) consists of disjoint smooth connected analytic surfaces S_n, $n \geq 0$, properly embedded in $\mathbf{R}^3$. For $(a, c, \lambda) \in S_n$, the eigenfunction has $2n$ non-real zeros. Each of these surfaces is homeomorphic to a punctured disc. Projection $\pi(a, c, \lambda) = (a, c)$ has the following properties: It is a 2-to-1 covering over some neighborhood of the a-axis, and for $a > a_0$, the preimage of every line $c = \mathrm{const}$ is compact and homeomorphic to a circle.*

Proof. We follow the same pattern as in the proof of Theorem 4.1. There are 6 Stokes sectors, $S_0, \ldots, S_5$, which we enumerate anticlockwise, beginning from the sector in the first quadrant.

If $f = w/w_1$ where w is a real eigenfunction and w_1 is a real linearly independent solution of the same equation, then f has asymptotic values $b_0, 0, b_1, \overline{b_1}, 0, \overline{b_0}$ in the sectors $S_0, \ldots, S_5$. Here $b_0 \neq b_1$, and b_0, b_1 must belong to $\mathbf{C}\backslash\mathbf{R}$.

If $c = 0$, we can choose w, w_1 with the additional symmetry with respect to the imaginary axis, which gives $b_0 = -\overline{b_1}$, so b_0 and b_1 belong to the same half-plane of $\mathbf{C}\backslash\mathbf{R}$. The same situation persists for all real c because b_0, b_1 depend continuously on c and never cross the real line. The real affine group acts on f by post-composition; this corresponds to the change of normalization of w and w_1. So we can always choose the normalization so that $b_1 = i$.

Notice that after this normalization condition $c = 0$ corresponds to $|b_0 - i/2| = 1/2$. See Remark 4.6 after the proof.

Consider the cell decomposition Φ of the Riemann sphere (the range of f) shown in Fig. 13. It consists of one vertex at ∞ and four disjoint loops around $\pm i$ and $b, \overline{b}$ that are interchanged by the symmetry.

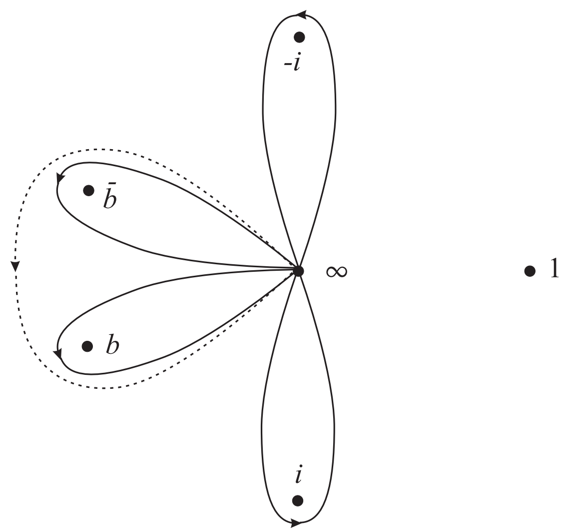


Figure 13. Cell decomposition Φ of the sphere (solid lines).

Now consider the cell decomposition Ψ_n of the plane (with labeled faces) shown in Fig. 14. It is locally similar to Φ, and depends on one integer parameter $n \geq 0$ which is the number of 0-labeled faces between the adjacent "ramification points".

As in Theorem 4.1, Nevanlinna theory gives for each $n \geq 0$ a family G_n of meromorphic functions f which have $2n$ non-real zeros and satisfy the Schwarz equation of the form

$$\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 = 2(z^4 - az^2 - cz - \lambda). \tag{4.10}$$

with real a, c, λ.

Classification result for symmetric trees with 6 faces in [19] ensures that all equations (4.1) having a solution with infinitely many real zeros and $2n$ non-real zeros are equivalent to equations which arise from our families G_n.

This also has an implication that there is "no monodromy" in our families G_n: when b traverses a loop around i, we return with the

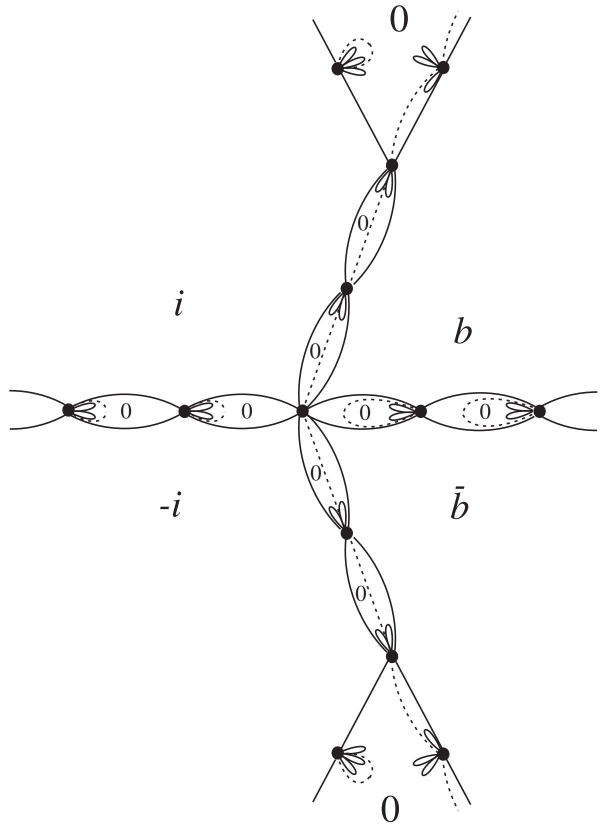


FIGURE 14. Cell decomposition Ψ_n for a quartic with $n = 2$ (solid lines).

same function f we started with. Indeed, in the process of continuous deformation the number of non-real zeros cannot change, and there is only one suitable cell decomposition Ψ_n for every n.

Thus our family G_n is homeomorphic to a punctured disc. Taking the coefficients a, c, λ of the Schwarzian defines an analytic embedding of G_n to $\mathbf{R}^3$. This is our surface S_n. The surfaces are disjoint and properly embedded for the same reasons as in the proof of Theorem 4.1.

To study the shape of these surfaces S_n in $\mathbf{R}^3$, we first notice that for $c = 0$, the eigenvalue problem obtained from (4.8) by rotation $z \mapsto iz$ is Hermitian. It follows that the intersection with the plane $S_n \cap \{(a, c, \lambda) : c = 0\}$ consists of the disjoint graphs of two analytic functions defined for all real a, and that $\lambda_n(a, 0) < \lambda_{n+1}(a, 0)$. Another simple property of the surface S_n is that it is symmetric with respect to change $c \mapsto -c$, which follows by changing $z \mapsto -z$ in the equation.

Now we study the asymptotic behavior of S_n for $a \to +\infty$. In the equation (4.8) we set $z = \varepsilon(\zeta - t),\ y(z) = w(\varepsilon(\zeta - t))$, where t satisfies

$$a - 6\varepsilon^2 t^2 = 0, \quad \text{and} \quad 4\varepsilon^6 t = 1, \tag{4.11}$$

and obtain

$$-y'' + (-\varepsilon^6 z^4 + z^3 + \alpha z + \mu)y = 0, \tag{4.12}$$

where

$$\alpha = 4\varepsilon^6 t^3 + 2\varepsilon^4 a t + c\varepsilon^3,$$

and

$$\mu = -\varepsilon^6 t^4 + a\varepsilon^4 t^2 - c\varepsilon^3 t + \varepsilon^2 \lambda.$$

Expressing t and a from equations (4.11) as functions of ε and substituting the result to the expression of α we obtain

$$a = (3/8)\varepsilon^{-10}, \quad c = \varepsilon^{-3}\alpha - (1/4)\varepsilon^{-15}, \tag{4.13}$$

$$\lambda = -21 \cdot 2^{-8}\varepsilon^{-20} + (\alpha/4)\varepsilon^{-8} + \mu\varepsilon^{-2}. \tag{4.14}$$

Consider the curves Γ_n from Theorem 4.1. It follows from their properties stated in Theorem 4.1, that for every n, there exists $\alpha_n = \max\{-\alpha : (\alpha, \lambda) \in \Gamma_n$, and $0 < \alpha_n < \infty$.

Suppose that $\alpha < \alpha_n$, and consider the curve in (a, c)-plane parametrized by (4.13). Equation (4.12) satisfies the conditions of Theorem 4.1 of Section 2 (the Stokes complex corresponding to this equation is shown in Fig. 1(a), rotated by 90°), and the sectors containing the normalizing rays are stable. We conclude that the spectrum of (4.12) tends to the spectrum of the cubic

$$-y'' + (z^3 + \alpha z + \mu)y = 0. \tag{4.15}$$

The spectrum of the cubic with parameter α has at least one eigenvalue μ^* which is real and such that the corresponding eigenfunction has $2n$ non-real zeros. As μ^* is an isolated point of this spectrum, and the spectrum of (4.12) is symmetric with respect to the real axis, we conclude that there is an eigenvalue μ of (4.12) which is real, and the corresponding eigenfunction has $2n$ non-real zeros.

To ensure that the number of non-real zeros does not change in the limit, we make an argument similar to that in the proof of Theorem 4.1, the degeneration of the cell decomposition Ψ_n is shown in Fig. 15.

We conclude that projection of our surface S contains a piece of the curve (4.13) for $\varepsilon \in (0, \varepsilon_0)$.

Now suppose that $\alpha > \alpha_n$. We claim that there are no points on S_n with $a \to \infty$ and (a, c) on the curve (4.13). Proving this by contradiction, we suppose that there is a sequence $(a_j, c_j, \lambda_j) \in S_n$ such that (a_j, c_j) belong to the curve (4.13). Then Theorem 2.3 implies that the

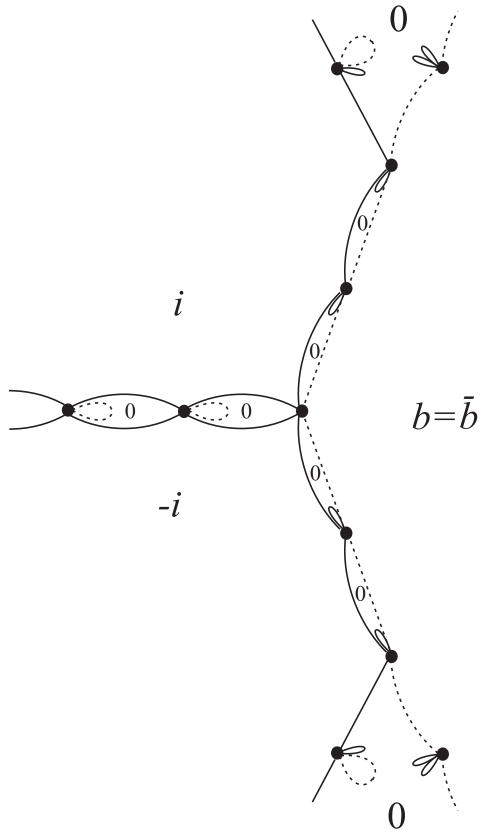


FIGURE 15. Limit cell decomposition with $n = 2$ (solid and dotted lines).

sequence μ_j related to the λ_j by (4.14), has the property that μ_j tends to a real eigenvalue μ^* of the cubic oscillator (4.15). Then the corresponding eigenfunction tends to an eigenfunction of the cubic with $2n$ non-real zeros. This is a contradiction because $\alpha > \alpha_n$, so our claim is proved.

So the projection of S_n on the plane (a, c) looks as a paraboloid $9c^2 - 4a^3 \leq 0$ when $a \to +\infty$. □

Fig. 16, which is taken from Trinh's thesis shows a section of the surfaces S_n by the plane $a = -9$. Similar pictures can be seen in [3, 13].

Computational evidence suggests that each S_n has the shape of an infinite funnel with a sharp end stretching towards $a = -\infty$. This end probably corresponds to $b \to i$, where b is the asymptotic value as in Figs. 13-14. $\lambda \to -\infty$ as $a \to -\infty$ on S_n as the picture in [13] suggests. For every real a_0 the section of S_n by the plane $a = a_0$ is

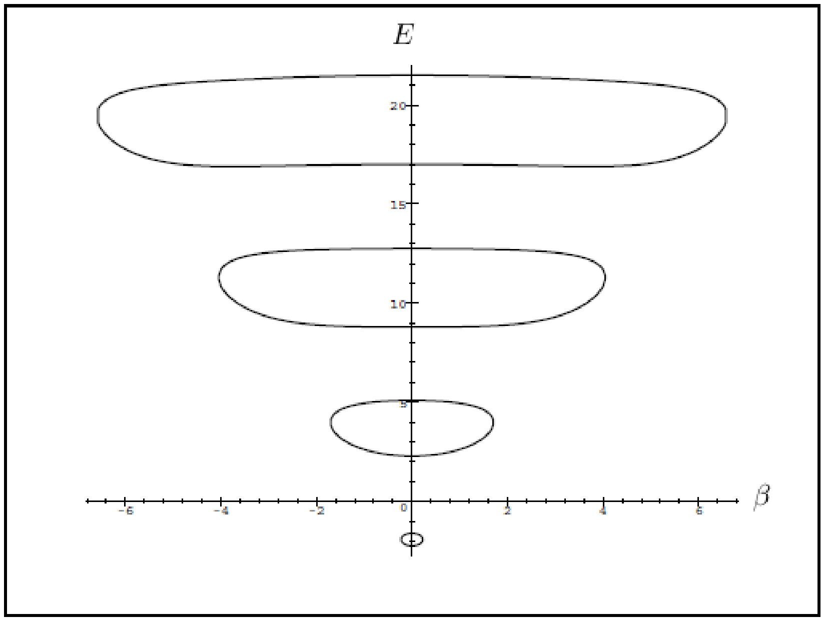


FIGURE 16. Section of the surfaces $S_n,\ n = 0, 1, 2, 3$ by the plane $a = -9$.

an oval that projects on the c-axis 2-to-1. We only proved that this section is compact for a large enough. For $n = 0, 1, 2, \ldots$, the funnels S_n are symmetric with respect to $c \mapsto -c$, S_{n+1} lies above S_n and S_{n+1} is wider than S_n.

Remark 4.6. In general, it is hard to say anything explicit on the correspondence between the parameters a, c in the potential and Nevanlinna parameter b. Some information on this correspondence can be extracted from symmetry and degeneration considerations. In the beginning of the proof of Theorem we noticed that the line $c = 0$ corresponds to the circle $|b - i/2| = 1/2$. We can determine now the sign of c for b inside and outside this circle. Degeneration used in the proof corresponds to convergence of b to a real non-zero point. Formula (4.13) shows that $c < 0$ when $\epsilon \to 0$. So negative c correspond to the exterior of the circle and positive c to its interior.

Now we state the result about the second PT-symmetric family of quartics.

Theorem 4.7. *The real QES part of the spectral locus of (4.9) consists of* $\lceil m/2 \rceil$ *simple disjoint analytic curves* $\Gamma^*_{m,n}$, $n = 0, \ldots, \lceil m/2 \rceil - 1$, *properly embedded curves which for $a > 0$ project onto the ray $a > 0$ 2-to-1. When $(a, \lambda) \in \Gamma_n^*$, the eigenfunction has $2n$ non-real zeros.*

The proof is completely similar to the proof of Theorem 4.1.

The problem of study of the whole real part of spectral locus of (4.9), as a two-parametric family with real m seems quite interesting and challenging. A picture of the spectral locus for $m = 3$ can be seen in [4].

Theorem 4.8. *Every equation of the form (4.1) with polynomial P of degree 4 which has a solution with $2n$ non-real zeros and infinitely many real zeros is equivalent by a real affine change of the independent variable to equation (4.8) with $(a, c, \lambda) \in S_n$.*

Every equation of the form (4.1) with polynomial P of degree 4 which has a solution with finitely many zeros is equivalent to (4.9) with $(a, \lambda) \in \Gamma_{m,n}^*$ by an affine change of the independent variable.

The proof is similar to that of Theorem 4.4.

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