Conjectured inequality for subharmonic functions

A. Eremenko

April 4, 2015

Let u_1, u_2, u_3 be three subharmonic functions in the plane, and

$$u_j(0) = 0, \quad 1 \leq j \leq 3.$$

For every point z the three numbers $u_j(z)$ can be ordered as $v_1(z) \leq v_2(z) \leq v_2(z)$, so we obtain three functions v_1, v_2, v_3. One can write formulas for them:

$$v_1(z) = \min_j u_j(z), \quad v_3(z) = \max_j u_j(z),$$

and

$$v_2(z) = \min\left\{\max\{v_1(z), v_2(z)\}, \max\{v_1(z), v_3(z)\}, \max\{v_2(z), v_3(z)\}\right\}.$$

Let

$$I(r, v) = \int_0^{2\pi} v(re^{i\theta}) d\theta, \quad B(r) = \max_{|z|=r} v_3(z).$$

Conjecture. For every three subharmonic functions u_j with $u_j(0) = 0$, we have

$$\sup_{r>0} I(r, v_2)/B(r) \geq 0.$$

There is also a stronger conjecture that

$$\limsup_{r \to 0} I(r, v_2)/B(r) \geq 0.$$

These conjectures have applications to value distribution of holomorphic curves. Functions arising in this application have an additional property that

$$u_j(z) \leq |z|^A, \quad \text{for all} \quad z,$$

with some positive constant A.

I can prove the conjecture for homogeneous subharmonic functions, that is

$$u_j(re^{i\theta}) = r^\rho h_j(\theta).$$

In this case, u_j is subharmonic if and only if h_j is trigonometrically convex,

$$h'' + \rho^2 h \geq 0$$

in the sense of distributions. Trigonometrically convex functions satisfy the inequality

$$h(\theta) + h(\theta + \pi/\rho) \geq 0. \tag{1}$$

Let

$$v_j = r^\rho H_j(\theta).$$

Then one can obtain the inequality

$$H_2(\theta) + H_2(\theta + \pi/\rho) \geq 0, \tag{2}$$

from which the conjecture follows by integration. To derive (2) from (1), one uses the following elementary fact.

Let $a_j, a'_j,\ 1 \leq j \leq 3$ be real numbers with the property

$$a_j + a'_j \geq 0.$$

Let $A_1 \leq A_2 \leq A_3$ be the rearrangement of a_1, a_2, a_3 in the increasing order, and A'_j are defined similarly. Then $A_2 + A'_2 \geq 0$.

As harmonic functions are approximately homogeneous when $a \to 0$, we obtain that the conjecture is true for harmonic functions.