A conjecture about families of subharmonic functions

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Let $F$ be an indexed set of subharmonic functions in the plane. This means that we have some set of indices $A$, finite or infinite, and a map from $A$ to the set of subharmonic functions in the plane, $\alpha \mapsto v_\alpha$.

Suppose that $F$ has the following properties:

1. $u(z) = \max\{v_\alpha(z) : \alpha \in A\}$ is subharmonic, and for every $z$,
   \[
   \text{card}\{\alpha \in A : v_\alpha(z) < u(z)\} \leq 2.
   \]
   In other words, at every point $z$ at most 2 functions of the family can be less than the maximum over the family.

2. For every two pairs of distinct indices $\alpha_1 \neq \alpha_2$ and $\alpha_3 \neq \alpha_4$, there exists $\alpha_5 \in A$, such that
   \[
   v_{\alpha_5} \leq \min\{\max\{v_{\alpha_1}, v_{\alpha_2}\}, \max\{v_{\alpha_3}, v_{\alpha_4}\}\}.
   \]

Such families of subharmonic functions arise in the problems about holomorphic curves.

Let $F$ be such a family. At every point $z$ the set $\{v_\alpha(z) : \alpha \in A\}$ contains at most 3 elements, which we order and denote by

\[
\begin{align*}
u_0(z) & \leq u_2(z) \leq u(z),
\end{align*}
\]

More precisely,

\[
u_0(z) = \min\{v_\alpha(z) : \alpha \in A\},
\]
and

\[ u_1(z) = \min_{\alpha \neq \beta} \max \{v_\alpha(z), v_\beta(z)\}. \]

**Problem.** Describe, which triples of functions \( u_0, u_1, u \) can occur if

\[ u_0 + u_1 + u = 0. \quad (1) \]

**Conjecture.** All triples \( u_0, u_1, u \) satisfying (1) are obtained by the following construction. Take a 3-sheeted Riemann surface spread over the plane, and let \( U \) be a harmonic function on this surface. Then \( u_0(z), u_1(z), u(z) \) are the ordered values of this 3-valued harmonic function over \( z \).

**Heuristic proof.** Let us break the plane into maximal regions \( G_j \) where \( u_0(z) < u_1(z) < u(z) \). In each such region, \( u_0, u_1, u \) are subharmonic, so they must be harmonic in view of (1). Consider the common boundary of two such regions \( G_1 \) and \( G_2 \). Suppose first that only two of \( u_0, u_1, u_3 \) collide on some piece \( \gamma \) of this common boundary. For example, suppose that \( u_0(z) = u_1(z) < u(z) \). There are functions \( v_1, v_2 \) in \( F \) such that

\[ u_0(z) = v_1(z), \quad z \in G_1 \]

and

\[ u_0(z) = v_2(z), \quad z \in G_2. \]

Let \( V \) be a small neighborhood of \( z_0 \in \gamma \). Then for every \( z \in V \) we have

\[ \{v_1(z), v_2(z), u(z)\} = \{u_0(z), u_1(z), u(z)\}. \]

So in \( V \) we have three subharmonic functions \( v_1, v_2 \) and \( u \) such that their sum is zero, therefore they are all harmonic.

Similar argument works when \( u_0(z) < u_1(z) = u(z) \) on \( \gamma \).

Now suppose that

\[ u_0(z) = u_1(z) = u(z), \quad z \in \gamma. \quad (2) \]

**Lemma.** In this case, \( u_1 \) is subharmonic in a neighborhood of \( \gamma \).

**Proof.** Let \( V \) be a neighborhood of a point on \( \gamma \) which intersects only \( G_1 \) and \( G_2 \). Let

\[ u_0(z) = v_1(z), \quad u_1(z) = v_2(z), \quad z \in G_1, \]

\[ u_0(z) = u_1(z) = u(z), \quad z \in \gamma. \]
and

\[ u_0(z) = v_3(z), \quad u_1(z) = v_4(z), \quad z \in G_2. \]

Then by property 2, there exists \( v_5 \in F \) such that

\[ v_5(z) \leq u_1(z), \quad z \in V. \]

Moreover we have \( v_5(z) = u_1(z), \quad z \in \gamma \). Then our lemma follows from Grishin’s lemma.

Continuing the proof under the assumption (2), and using the notation of the proof of the Lemma. Three subharmonic functions \( v_1, v_3, u_1 \) have the property that

\[ v_1 + v_3 + u_1 \leq u_0 + u_1 + u = 0 \quad \text{in} \quad V, \]

and on the other hand, this sum equals 0 on \( \gamma \). Therefore we have equality and for every \( z \in V \), so our functions are harmonic, and for every \( z \in V \) the set \( \{v_1(z), v_2(z), u_1(z)\} \) coincides with the set \( \{u_0(z), u_1(z), u(z)\} \).

Let \( E \) be the closed set where the boundaries of more than two \( G_j \) intersect. We proved that every point of \( \mathbb{C} \setminus E \) has a neighborhood \( V \) where there exist three harmonic functions \( v_1, v_2, v_3 \) such that for every \( z \in V \)

\[ \{v_1(z), v_2(z), v_3(z)\} = \{u_0(z), u_1(z), u(z)\}. \]

So there exists a 3-sheeted Riemann surface spread over \( \mathbb{C} \setminus E \), and a harmonic function on it with the required property. If \( E \) consists of isolated points, this would complete the proof by the removable singularity theorem.