A conjecture about families of subharmonic functions

A. Eremenko

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Let F be an indexed set of subharmonic functions in the plane. This means that we have some set of indices A, finite or infinite, and a map from A to the set of subharmonic functions in the plane, $\alpha \mapsto v_\alpha$.

Suppose that F has the following properties:

1. $u(z) = \max\{v_\alpha(z) : \alpha \in A\}$ is subharmonic, and for every z,

$$\text{card}\{\alpha \in A : v_\alpha(z) < u(z)\} \leq 2.$$

In other words, at every point z at most 2 functions of the family can be less than the maximum over the family.

2. For every two pairs of distinct indices $\alpha_1 \neq \alpha_2$ and $\alpha_3 \neq \alpha_4$, there exists $\alpha_5 \in A$, such that

$$v_{\alpha_5} \leq \min\left\{\max\{v_{\alpha_1}, v_{\alpha_2}\}, \max\{v_{\alpha_3}, v_{\alpha_4}\}\right\}.$$

Such families of subharmonic functions arise in the problems about holomorphic curves.

Let F be such a family. At every point z the set $\{v_\alpha(z) : \alpha \in A\}$ contains at most 3 elements, which we order and denote by

$$u_0(z) \leq u_2(z) \leq u(z).$$

More precisely,

$$u_0(z) = \min\{v_\alpha(z) : \alpha \in A\},$$

and

$$u_1(z) = \min \max_{\alpha \neq \beta} \{v_\alpha(z), v_\beta(z)\}.$$

Problem. Describe, which triples of functions u_0, u_1, u can occur if

$$u_0 + u_1 + u = 0. \tag{1}$$

Conjecture. All triples u_0, u_1, u satisfying (1) are obtained by the following construction. Take a 3-sheeted Riemann surface spread over the plane, and let U be a harmonic function on this surface. Then $u_0(z), u_1(z), u(z)$ are the ordered values of this 3-valued harmonic function over z.

Heuristic proof. Let us break the plane into maximal regions G_j where $u_0(z) < u_1(z) < u(z)$. In each such region, u_0, u_1, u are subharmonic, so they must be harmonic in view of (1). Consider the common boundary of two such regions G_1 and G_2. Suppose first that only two of u_0, u_1, u_3 collide on some piece γ of this common boundary. For example, suppose that $u_0(z) = u_1(z) < u(z)$. There are functions v_1, v_2 in F such that

$$u_0(z) = v_1(z), \quad z \in G_1$$

and

$$u_0(z) = v_2(z), \quad z \in G_2.$$

Let V be a small neighborhood of $z_0 \in \gamma$. Then for every $z \in V$ we have

$$\{v_1(z), v_2(z), u(z)\} = \{u_0(z), u_1(z), u(z)\}.$$

So in V we have three subharmonic functions v_1, v_2 and u such that their sum is zero, therefore they are all harmonic.

Similar argument works when $u_0(z) < u_1(z) = u(z)$ on γ.

Now suppose that

$$u_0(z) = u_1(z) = u(z), \quad z \in \gamma. \tag{2}$$

Lemma. In this case, u_1 is subharmonic in a neighborhood of γ.

Proof. Let V be a neighborhood of a point on γ which intersects only G_1 and G_2. Let

$$u_0(z) = v_1(z),\ u_1(z) = v_2(z), \quad z \in G_1,$$

and

$$u_0(z) = v_3(z),\ u_1(z) = v_4(z), \quad z \in G_2.$$

Then by property 2, there exists $v_5 \in F$ such that

$$v_5(z) \leq u_1(z), \quad z \in V.$$

Moreover we have $v_5(z) = u_1(z),\ z \in \gamma$. Then our lemma follows from Grishin's lemma.

Continuing the proof under the assumption (2), and using the notation of the proof of the Lemma. Three subharmonic functions v_1, v_3, u_1 have the property that

$$v_1 + v_3 + u_1 \leq u_0 + u_1 + u = 0 \quad \text{in} \quad V,$$

and on the other hand, this sum equals 0 on γ. Therefore we have equality and for every $z \in V$, so our functions are harmonic, and for every $z \in V$ the set $\{v_1(z), v_2(z), u_1(z)\}$ coincides with the set $\{u_0(z), u_1(z), u(z)\}$.

Let E be the closed set where the boundaries of more than two G_j intersect. We proved that every point of $\mathbf{C}\backslash E$ has a neighborhood V where there exist three harmonic functions v_1, v_2, v_3 such that for every $z \in V$

$$\{v_1(z), v_2(z), v_3(z)\} = \{u_0(z), u_1(z), u(z)\}.$$

So there exists a 3-sheeted Riemann surface spread over $\mathbf{C}\backslash E$, and a harmonic function on it with the required property. If E consists of isolated points, this would complete the proof by the removable singularity theorem.