An inequality for polynomials and potentials

Alexandre Eremenko

December 22, 2010

Let P be a monic polynomial of degree d and $E = \{z : |P(z)| = 1\}$. It is known [1] that

$$\max_E |P'(z)| \leq 2^{1/n-1} d^2,$$

with equality only for Chebyshev polynomials; the fact conjectured by Erdös.

In this paper we prove the lower estimate

$$\max_E |P'(z)| \geq d, \tag{1}$$

with equality only for $R(z) = z^d$.

Proof of (1). The set E is of capacity 1. The set $E' = \{z : |P'(z)/d| \leq 1\}$ is also of capacity 1. If E intersect the complement of E' then (1) holds.

If not, $E \subset E'$. Both sets are bounded by finitely many piecewise analytic curves and each component of E or E' is simply connected. As they have the same capacity, we conclude that $E = E'$. This implies that $|P|^{1/d} = |P'|^{1/(d-1)}$, and we conclude that $P(z) = z^d$.

This can be generalized as follows:

Theorem. *Let E be a regular compact subset of the plane of capacity 1. Let u be the Green function of $D = \overline{\mathbf{C}} \backslash E$ with the pole at infinity. Then*

$$\sup_D |\mathrm{grad}\, u| \geq 1.$$

Corollary. *Let $f(z) = 1/z + O(1), z \to 0$, be a univalent function in the unit disc $\mathbf{U}$. Then*

$$\inf_{\mathbf{U}} |f'| \leq 1,$$

with equality only if $f(z) = z + c$.

References

[1] A. Eremenko, L. Lempert, An extremal problem for polynomials, Proc. AMS, 122, 1 (1994) 191–193.