Proof that $t \mapsto \exp(it) : \mathbf{R} \to \mathbf{T}$ is surjective

Another proof is in Ahlfors, p. 45, or Whittaker-Watson, vol. 1, Appendix. All other authors seem to rely on the facts about trigonometric functions "proved" in high school.

A *topological group* G is a set with a group structure and topology, so that the group operations are continuous. This means that the maps $G \times G \to G, (x, y) \mapsto xy$ and $G \to G, x \mapsto 1/x$ are continuous (first of them, with respect to product topology). *Morphisms* of topological groups are defined as continuous homomorphisms. Examples of topological groups are $(\mathbf{Z}, +)$ with discrete topology and $(\mathbf{R}, +), (\mathbf{C}, +), (\mathbf{C}^*, .), \mathbf{T}$ with their natural topologies.

Proposition. *If G is a topological group, and $H \subset G$ a subgroup, which contains a neighborhood of unity, then H is open and closed, that is H is a connected component of G.*

Proof. Let U be this neighborhood of the unity. If $x \in H$ then $xU := \{xy : y \in U\}$ is a neighborhood of x, which is contained in H. So H is open. Now suppose that $x \in \bar{H}$. Then $xU \cap H \neq \emptyset$, and we choose an element $y \in xU \cap H$. Then $x = yz^{-1}$ for some $z \in U \subset H$, thus $x \in H$. This shows that H is closed. □

In the second lecture we defined a morphism of topological groups $(\mathbf{R}, +) \to \mathbf{T}, \quad \mathbf{t} \mapsto \exp(\mathbf{it})$. If $H \subset \mathbf{T}$ is the image of this morphism, then H is a topological subgroup of $\mathbf{T}$. We age going to prove that $H = \mathbf{T}$.

Lemma. *There is a neighborhood U of 1 in $\mathbf{T}$, such that the map $U \to \mathbf{R}$, $\mathbf{z} \mapsto \Im\mathbf{z}$, is a homeomorphism onto its image.*

Proof. The equation of the unit circle in $\mathbf{R^2}$ is $x^2 + y^2 = 1$. For each $|y| < 1/2$ this equation, with respect to x, has exactly one positive solution. Thus we can take $U = \{z \in \mathbf{T} : \Re\mathbf{z} > \mathbf{0}, |\Im\mathbf{z}| < \mathbf{1/2}\}$. □

Let V be the component of $\exp^{-1}(U)$, which contains the origin. The map $V \to \mathbf{R}$, $\mathbf{t} \mapsto \Im\exp(\mathbf{it}) = \sin\mathbf{t}$ is differentiable with positive derivative at 0. So there exists a neighborhood V' of 0 in $\mathbf{R}$, such that the restriction $\sin : V' \to \mathbf{R}$ is a homeomorphism onto its image. It follows that $t \mapsto \exp(it) : V' \to \mathbf{T}$ is a homeomorphism onto the image, thus the image U' is a neighborhood of 1 in $\mathbf{T}$.

We have seen, that the subgroup $H \subset \mathbf{T}$ contains a neighborhood of the unity, namely U'. Thus by the Proposition $H = \mathbf{T}$, because $\mathbf{T}$ is connected.