Metrics of constant positive curvature with conic singularities

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Abstract

We consider conformal metrics of constant curvature 1 on a Riemann surface, with finitely many prescribed conic singularities and prescribed angles at these singularities.

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1. Introduction.

Let S be a compact Riemann surface and $A = \{a_1, \ldots, a_n\} \subset S$ a finite set. A conformal Riemannian metric is given by the length element $\rho(z)|dz|,\ \rho > 0$, and the curvature of this metric is

$$\kappa = -\frac{\Delta \log \rho}{\rho^2}.$$

We assume that the curvature is constant on $S \backslash A$ while at the points a_j the metric has *conic singularities*:

$$\rho(z) \sim c|z - a_j|^{\alpha_j - 1}, \quad z \to a_j.$$

Positive numbers α_j are the *angles* at the singularities (*we measure angles in turns; one turn* $= 2\pi$ *radians*).

When there are no singularities, a classical result says that on each Riemann surface there is a metric with $\kappa = 1$ when S is the sphere, $\kappa = 0$ when S is a torus, and $\kappa = -1$ for all other Riemann surfaces. This metric is unique except for $\kappa = 0$ when it can be multiplied by an arbitrary positive number.

The problem we discuss here is how to understand (describe, classify) such metrics with prescribed singularities and angles at the singularities. This problem has been studied by a large variety of methods: PDE [31, 32, 33, 34, 26, 28, 37], non-linear functional analysis [1, 3], analytic theory of linear ODE [9, 10, 18, 27], complex analysis [11, 15] geometry [15, 29, 30, 28], elliptic functions and modular forms [12, 26], and holomorphic dynamics [2].

The simplest examples of such metrics are obtained from polygons. An n-gon is a simply connected bordered Riemann surface equipped with a Riemannian metric of constant curvature κ, and such that the boundary consists of n geodesic arcs meeting at n corners. Two n-gons are called conformally equivalent if there is a conformal homeomorphism between them sending corners to corners. Gluing a polygon to its mirror image we obtain a sphere with metric of constant curvature with conic singularities at the corners. When $\kappa = 0$, the Gauss–Bonnet theorem implies that the sum of the interior angles at the corners is $n - 2$, and the Christoffel–Schwarz formula tells us that for prescribed angles and prescribed conformal class, there is one polygon, up to similarity. The situation is similar in hyperbolic geometry, but it is much more complicated in spherical geometry: there are additional restrictions on the angles, and when they are satisfied the polygon with prescribed angles in a given conformal class may be not unique. Metrics on the sphere coming from polygons are characterized by the property that there is a circle on the sphere which contains all singularities, and the metric is symmetric with respect to this circle.

A complete solution of our problem is known when $\kappa \leq 0$. The Gauss–Bonnet theorem implies that on a surface of Euler characteristic $\chi(S)$,

$$\chi(S) + \sum_{j=1}^{n} (\alpha_j - 1) = \frac{1}{2\pi}(\text{total integral curvature}), \tag{1}$$

so the expression in the LHS has the same sign as κ.

When $\kappa \leq 0$, this is a necessary and sufficient condition for the existence of the metric; it is unique when $\kappa < 0$ and unique up to a constant factor when $\kappa = 0$.

This result goes back to Picard, [31, 32, 33, 34]. Modern proofs can be found in [24, 37]. All these proofs are based on the consideration of solutions of the differential equation $\Delta u + \kappa e^{2u} = 0$ on $S \backslash A$ with prescribed behavior at the singularities. Here $u = \log \rho$. This equation can be also written as

$$\Delta u + \kappa e^{2u} = 2\pi \sum_{j=1}^{n} (\alpha_j - 1)\delta_{a_j} \quad \text{on} \quad S.$$

On the other hand, the problem with $\kappa > 0$ is wide open, and it is the subject of this paper.

From now on we assume that $\kappa = 1$.

2. Developing map and a general conjecture.

A surface of curvature 1 is locally isometric to a region on the standard sphere which we denote by $\overline{\mathbf{C}}$. (The length element of the metric on $\overline{\mathbf{C}}$ is $2|dz|/(1 + |z|^2)$.) This local isometry can be analytically continued along any path not passing through the singularities so we have a multivalued *developing map*

$$f : S\backslash A \to \overline{\mathbf{C}}.$$

As a local isometry, this map is holomorphic, and its monodromy is a subgroup of the group of rotations of the sphere $SO(3) = PSU(2) = SU(2)/\{\pm I\}$. Near the singularities we have

$$f(z) \sim c(z - a_j)^{\alpha_j}. \tag{2}$$

Conversely, any multivalued locally biholomorphic function f on $S\backslash A$ with $PSU(2)$ monodromy and satisfying (2) near the points $a_j \in A$ is a developing map of some metric on the sphere of curvature 1 with conic singularities at a_j with angles α_j. The metric is recovered from f by the formula

$$\rho(z) = \frac{2|f'|}{(1 + |f|^2)}.$$

Developing map is not unique: two developing maps f and g define the same metric if $f = \phi(g)$ where $\phi \in PSU(2)$.

So classification of metrics is equivalent to classification of developing maps modulo composition with rotations.

For a developing map f it can happen that $\phi(f)$ with $\phi \in PSL(2, \mathbf{C})$ is also a developing map even when ϕ is not a rotation. This happens if and only if ϕ conjugates the monodromy group Γ of f to a subgroup of $PSU(2)$.

It is easy to see that this can happen with $\phi \notin SU(2)$ if and only if Γ is isomorphic to a subgroup of the unit circle $O(2)$.

In this case, the monodromy and the metric are called *co-axial.* Existence of co-axial metrics justifies the following definition:

Two metrics are called equivalent if their developing maps are obtained from each other by post-composition with a linear-fractional transformation.

Equivalence classes are 3-parametric when Γ is trivial, one parametric when Γ is a non-trivial subgroup of the unit circle, and consist of one element in all other cases.

Now we can state a general conjecture.

Conjecture 1. *For any compact Riemann surface S, and any prescribed singularities a_j and angles α_j, there are finitely many equivalence classes of metrics of curvature 1 on S with these angles at these singularities.*

Even in the simplest cases there can be more than one class, in sharp contrast with the case of non-positive curvature. The simplest example of non-uniqueness is given in Section 6. Conjecture 1 has been proved for the case when S is the sphere with 4 singularities [10].

If the general conjecture is true, the next question is

Question 1. *How many equivalence classes of metrics exist on a compact Riemann surface with prescribed angles and singularities?*

The proof in [10] is not constructive and gives no upper estimate.

3. General restrictions on the angles.

Here we address the question what angles α_j can occur (without prescribing the position of the singularities a_j or conformal type of S). The Gauss-Bonnet theorem for the sphere gives

$$2 + \sum_{j=1}^{n} (\alpha_j - 1) > 0. \tag{3}$$

Unlike in the case $\kappa \leq 0$, in the case when S is the sphere, there is another condition, which is called the *closure condition*:

$$d_1(\alpha - \mathbf{1}, \mathbf{Z}_o^n) \geq 1, \tag{4}$$

where $\alpha - \mathbf{1} = (\alpha_1 - 1, \dots, \alpha_n - 1)$, $\mathbf{Z}_o^n$ is the odd integer lattice (the set of points in $\mathbf{R}^n$ whose coordinates are integers whose sum is odd), and d_1 is the ℓ_1 distance.

Theorem 1. (Mondello and Panov [29]) *Conditions (3) and (4) are necessary for existence of a metric of curvature* 1 *on the sphere with angles α_j and some (unspecified) singularities a_j.*

Conditions (3) and (4) with strict inequality are sufficient.

Equality in condition (4) can only hold for metrics with co-axial monodromy.

Several special cases of (4) were known before [29], usually they were stated in different forms.

Possible angles of co-axial metrics on the sphere are described in [11].

To state the result, let us call a vector $\alpha = (\alpha_1, \dots, \alpha_n)$ with positive coordinates admissible if there exists a co-axial metric with angles α_j, and suppose without loss of generality that $\alpha_1, \dots, \alpha_m$ are not integers, while $\alpha_{m+1}, \dots, \alpha_n$ are integers.

Theorem 2. *For α to be admissible it is necessary that:*

(i) there exist a choice of signs $\epsilon_j \in \{\pm 1\}$ and an integer $k' \geq 0$ such that

$$\sum_{j=1}^{m} \epsilon_j \alpha_j = k',$$

(ii) the integer

$$k'' := \sum_{j=m+1}^{n} \alpha_j - n - k' + 2 \quad \textit{is even and non-negative.}$$

If the numbers

$$\mathbf{c} = (c_1, \dots, c_q) := (\alpha_1, \dots, \alpha_m, \underbrace{1, \dots, 1}_{k'+k'' \text{ times}})$$

are incommensurable, then (i) and (ii) are also sufficient.

(iii) If $\mathbf{c} = \eta \mathbf{b}$, where $\eta \neq 0$ and b_j are integers, then there is an additional necessary condition

$$2 \max_{m+1 \leq j \leq n} \alpha_j \leq \sum_{j=1}^{q} |b_j|, \tag{5}$$

and in this case the three conditions (i), (ii) and (5) are sufficient.

Parameter count shows that in the case of co-axial monodromy, if the number of non-integer angles m is greater than 2, the positions of the singularities cannot be arbitrarily prescribed, see [6, 7] about this.

These results give a complete description of possible angles at the conic singularities for metrics of curvature 1 on the sphere.

The situation on surfaces of higher genus is simpler:

Theorem 3. (Mondello and Panov [30]) *For any even* $\chi \leq 0$ *and any* $\alpha_j,\ j = 1, \ldots, n$ *satisfying*

$$\chi + \sum_{j=1}^{n} (\alpha_j - 1) > 0$$

there exists a compact Riemann surface S *of Euler characteristic* χ *and a metric of curvature* 1 *on* S *with conic singularities with angles* α_j.

A generalization of co-axial monodromy is the *dihedral monodromy*; it is characterized by the condition that all monodromy transformations preserve a set of two diametrally opposite points on the sphere. Possible angles of co-axial and dihedral metrics on surfaces of arbitrary genus are described in [21].

4. Fuchsian differential equations.

The monodromy of the developing map is a subgroup of the group of linear fractional transformations $PSL(2, \mathbf{C})$, therefore the Schwarzian derivative

$$F := \frac{f'''}{f} - \frac{3}{2}\left(\frac{f''}{f}\right)^2$$

is single-valued and defines a quadratic differential $F(z)dz^2$ holomorphic on $S \backslash A$, having double poles at the singularities.

We conclude that $f = w_1/w_2$, where w_1 and w_2 are linearly independent solutions of a differential equation

$$w'' + Pw' + Qw = 0, \tag{6}$$

where

$$F = -P' - P^2/2 + 2Q,$$

and there is some freedom in this choice. Changing P results in multiplication of w_1 and w_2 by a common factor. For example we can choose $P = 0, Q = F/2$, so that (6) becomes

$$w'' + (F/2)w = 0. \tag{7}$$

From the asymptotics of f at the points a_j we conclude that equation (6) has regular singularities, and the exponent differences at each singularity a_j is $\pm\alpha_j$. Moreover, the projective monodromy of this equation must be conjugate to a subgroup of $PSU(2)$.

We have a bijective correspondence between equivalence classes of metrics of curvature 1 with conic singularities at a_j with angles α_j and differential equations (7) with singularities at a_j, exponent differences $\pm\alpha_j$ and $PSU(2)$ projective monodromy up to conjugacy.

The set of equations (7) on a given Riemann surface of genus g with prescribed singularities and exponent differences depends on $3g + n - 3$ parameters which are called *accessory parameters*. These parameters have to be chosen so that the projective monodromy is conjugate to a subgroup of $PSU(2)$. So our main question is equivalent to the following

For equation (7) with prescribed singularities and prescribed real exponent differences, how many choices of accessory parameters exist so that the monodromy of this equation is conjugate to a subgroup of $PSU(2)$?

In the simplest cases of the sphere with four singularities and a torus with one singularity we have one (complex) equation on one accessory parameter.

Below, in sections 6–10, we describe all cases when the answer to the main questions stated in Section 3 is known. But first we state some general results.

Theorem 4. (Feng Luo, [27]). *Let Q be the fibration over the Teichmüller space $T_{g,n}$ of surfaces S of genus g with n punctures whose fiber at a point S is the space of quadratic differentials with at most double poles at the punctures. Let*

$$p : Q \to \mathrm{Hom}\,(\pi_1(S), PSL(2, \mathbf{C}))\,/PSL(2, \mathbf{C})$$

be the monodromy map. Then p is locally biholomorphic at every point x where all the exponent differences are not integers, and $p(x)$ is a smooth point of $\mathrm{Hom}\,(\pi_1(S), PSL(2, \mathbf{C}))\,/PSL(2, \mathbf{C})$.

The space of projective monodromy representations with fixed traces of the n generators corresponding to the punctures depends on $6g + 2n - 6$

complex parameters. These parameters are traces of certain elements of the projective monodromy group. The condition that monodromy is unitarizable is that all traces are real and satisfy certain inequalities. Thus the restriction of p in Theorem 4 on the set of equations with fixed singularities and angles is a holomorphic immersion of a complex manifold of dimension $3g+n-3$ to a complex manifold of dimension $6g+2n-6$. The condition of unitarizability imposes $6g+2n-6$ *real* equations. The main part of Conjecture 1 is that the set of solutions of these equations is discrete. There are "explicit" expressions of the derivative of p in [25, 8] but they are too complicated to obtain the necessary conclusion. So far, the discreteness of the set of solutions has been proved only in the simplest cases $(g,n)=(0,4)$ and $(g,n)=(1,1)$ when we have two real equations on one complex variable [10].

The problem is similar to the problem investigated by Klein and Poincaré in their attempts to prove the Uniformization theorem. They tried to show that one can choose accessory parameters so that the resulting monodromy group is Fuchsian (this is also described by reality of traces plus some inequalities). Eventually the Uniformization theorem was proved by other methods. The approach of Klein and Poincare using a Fuchsian equation has been recently completed in the book [35]. But the proof of the required transversality property of the monodromy map uses the properties of the hyperbolic metric [35, VIII.5.3] in an essential way.

Following [30] we denote by $\mathrm{Sph}_g(\alpha_1,\ldots,\alpha_n)$ the moduli space of metrics of curvature 1 on a surface S of genus g, with conic singularities with angles α_j. It follows from Theorem 4, that when none of the α_j are integers, and $2g+n-2>0$, $\mathrm{Sph}_g(\alpha_1,\ldots,\alpha_n)$ is a real analytic manifold of dimension $6g-6+2n$.

A Riemannian metric defines the conformal structure on $S\backslash A$, so we have the *forgetful map*

$$\Phi:\mathrm{Sph}_g(\alpha_1\ldots,\alpha_n)\to M_{g,n},$$

where $M_{g,n}$ is the moduli space of n-punctured Riemann surfaces of genus g.

To state the main results of [30] we need the following definitions.

$$\mathrm{Crit}_\alpha=\{\|\alpha_I\|_1-\|\alpha_{I^c}\|_1+2b:I\subset\{1,\ldots,n\},b\in\mathbf{Z}_{\geq 0}\},$$

where $\|\alpha_I\|_1=\sum_{j\in I}\alpha_j$ and $I^c=\{1,\ldots,n\}\backslash I$. Then the *non-bubbling parameter* is defined by

$$NB_{g,\alpha}=d_{\mathbf{R}}\left(\chi(S\backslash A),\mathrm{Crit}_\alpha\right),$$

where d_R is the ordinary distance between subsets of real line.

Theorem 5. (Mondello and Panov [30]) *If $NB_{g,\alpha} > 0$ then the forgetful map Φ is proper.*

This means that under the condition $NB_{g,\alpha} > 0$ the metric cannot degenerate unless the conformal modulus degenerates. On the other hand without this non-bubbling condition, such a degeneration is possible, see section 10. So to prove Conjecture 1 in section 2 under the condition that $NB_{g,\alpha} > 0$ it remains to prove that the set of accessory parameters corresponding to fixed position of singularities and fixed angles is discrete.

Another result in Mondello and Panov [30] is that the moduli space $\mathrm{Sph}_g(\alpha_1, \dots, \alpha_n)$ may be disconnected for some choice of α_j.

Topology of the the moduli space $\mathrm{Sph}_1(\alpha_1)$ of tori with one singularity has been described in [17]. When α is not an odd integer, denote $m = \lfloor(\alpha_1+1)/2\rfloor$. Then $M_1(\alpha_1)$ is a connected surface of genus $\lfloor(m^2-6m+12)/12\rfloor$ with m punctures.

When α_1 is an odd integer, the metrics are coaxial, and the set of equivalence classes has $\lceil m(m+1)/6\rceil$ connected components each of which is an open disk.

When α_1 is an even integer, $M_1(\alpha_1)$ has a natural complex analytic structure (it is an algebraic curve), the forgetful map is complex analytic (algebraic) and its degree is $\alpha_1/2$.

5. Topological degree.

One can use the topological (Leray–Schauder) degree of the equation

$$\Delta \log \rho + \rho^2 = 2\pi \sum_{j=0}^{n-1} (\alpha_j - 1)\delta_{a_j} \tag{8}$$

for the density ρ of the metric to obtain a lower estimate of the number of solutions.

Theorem 6.[1] *Let S be a compact surface of genus $g > 0$. Suppose that the angles $\alpha_1, \dots, \alpha_n$ satisfy $\alpha_j > 1,\ 1 \leq j \leq n$,*

$$\chi(S) + \sum_{j=1}^{n} (\alpha_j - 1) > 2 \min\{\alpha_1, \dots, \alpha_n, 1\},$$

and
$$\sum_{j\in I}\alpha_j-\sum_{j\notin I}\alpha_j\neq 2k-2+n+2g,\quad k\in\mathbf{Z}. \tag{9}$$
Then there is at least one metric on S with conic singularities at any given points with angles α_j.

For example, there is always a metric on torus with a single singularity where the angle is $2\pi\alpha$ and α is not an odd integer. On the other hand, we know from the previous section that when $\alpha = 3$ such metric may exist or not, depending on the torus.

Condition (9) coincides with the condition $NB_{g,\alpha} > 0$ in Theorem 5. Chen and Lin [3] actually computed the degree and included the case $g = 0$. They define the generating function

$$g(x)=(1+x+x^2+\ldots)^{-\chi(S)+n}\prod_{j=1}^{n}\left(1-x^{\alpha_j}\right),$$

where $\chi(S) = 2 - 2g$. Suppose that

$$g(x)=1+b_1x^{n_1}+b_2x^{n_2}+\ldots+b_kx^{n_k}+\ldots,$$

(this is the definition of b_k).

Theorem 7. *Let d be the Leray–Schauder degree of (8). Define k by the inequalities*
$$2n_k<\chi(S)+\sum_j(\alpha_j-1)<2n_{k+1},$$
(this is well defined if (9) holds). Then

$$d=\sum_{j=0}^{k}b_j.$$

The lower estimate of the number of metrics which follows from degree computation is sometimes best possible.

Example. For a torus with one singularity with angle α, the degree is defined when α is not an odd integer, and it is equal to m where $2m$ is the closest even integer to α. When $\alpha = 2m$, the forgetful map has m preimages for every generic point.

We will see in Section 9 that the lower estimate of the number of metrics which follows from degree computation is sometimes best possible. This is not surprising since sometimes the forgetful map is complex analytic.

6. All angles on the sphere are less than 1.

Theorem 8. (Luo and Tian, [28]) *A metric of curvature* 1 *on the sphere with prescribed singularities a_j and angles $\alpha_j < 1$ exists if an only if*

$$0 < 2 + \sum_{j=1}^{n}(\alpha_j - 1) \leq 2 \min_j\{\alpha_j\}. \tag{10}$$

Such a metric is unique.

The LHS inequality is (3) and the RHS inequality is equivalent to (4) for this case. Earlier Troyanov [37] proved sufficiency of (10). Luo and Tian combined PDE arguments with geometry by considering a convex polytope in S^3 associated with the metric in question.

7. All angles on the sphere are integers.

In this case, the monodromy is trivial, and the developing map f is a rational function. The singularities are the critical points of this rational function of multiplicity $\alpha_j - 1$. From the Riemann–Hurwitz relation we obtain that

$$2 + \sum_{j=1}^{n}(\alpha_j - 1) = 2d,$$

where d is the degree of f, and another restriction is $\alpha_j \leq d$ for all j, which can be obtained from (4). So the main problem is reduced to the question: *how many equivalence classes of rational functions exist with prescribed critical points of prescribed multiplicities?*

The answer is known for generic position of critical points: it is the Kostka number $K(\alpha_1 - 1, \ldots, \alpha_n - 1)$ which can be defined as follows. Consider a rectangular diagram of size $2 \times (2d - 2)$, and fill it with numbers $1, 2, \ldots, n$, so that the number k is used $\alpha_k - 1$ times, and so that the entries are strictly increasing in columns, and non-decreasing in rows. The number of obtained tables is the Kostka number K.

This result is due to Scherbak [36]. When $\alpha_j = 2$ for all k, the Kostka number is the Catalan number:

$$\frac{(2d-2)!}{d(d-1)!},$$

and in this special case the result was obtained by L. Goldberg [22].

It follows from the results in [22, 36] that there is always at least one metric and at most K equivalence classes of them.

The simplest example of non-uniqueness is obtained by considering a rational function of degree 3, there are two such non-equivalent functions sharing the position of their 4 simple critical points. There are only two exceptional positions of critical points, when there is only one class (see [22]).

An interesting special case is when all critical points of f lie on a circle (a circle on the Riemann sphere is a set whose points are fixed by an anti-conformal involution; this notion does not depend on the metric). In this case, the number of classes of metrics is exactly equal to the Kostka number, for any location of singularities on a circle [14]. Moreover, each class contains a metric which is symmetric with respect to this circle.

So in the case of integer angles our main question has a satisfactory solution.

8. All but two angles on the sphere are integers.

We mention that a metric on the sphere cannot have one non-integer angle (this is seen from the monodromy consideration).

In the case of two non-integer angles (and all the rest integers) the monodromy is co-axial, and the developing map has the form $z^\beta g(z)$ where g is a rational function. The result of Scherbak mentioned in the previous section has been generalized to this case in [15].

Theorem 9. *Let $a_1, \ldots, a_n$ and $\alpha_1, \ldots, \alpha_n$ be given, where α_1, α_2 are not integers, while $\alpha_3, \ldots, \alpha_n$ are integers. Then there is always at least one, and at most $E(\alpha_1, \ldots, \alpha_n) < \infty$ classes of metrics of curvature 1 on the sphere with these singularities and angles. For generic singularities equality holds. In the case when $a_1, a_2, \ldots, a_n$ lie on a circle, in this order, we have an equality, and each class contains a metric which is symmetric with respect to this circle.*

The number E can be expressed in terms of Kostka numbers, but the expression is somewhat complicated.

9. All but three angles on the sphere are integers.

First we mention an early result which completely solves the problem for the sphere with three singularities [9, 19]. In this case the location of singularities is of course irrelevant, and existence of the metric can be obtained from the recent results [29], [11] described in section 1. To this one can add that there is always a unique class of such metrics.

In the case when only three singularities have non-integer angles, say $\alpha_1, \alpha_2, \alpha_3$, the Gauss-Bonnet and the closure conditions can be written as one inequality,

$$\cos^2 \pi\alpha_1 + \cos^2 \pi\alpha_2 + \cos^2 \pi\alpha_3 + 2(-1)^\sigma \cos \pi\alpha_1 \cos \pi\alpha_2 \cos \pi\alpha_3 \leq 1, \quad (11)$$

where

$$\sigma = \sum_{j=4}^{n} (\alpha_j - 1).$$

Theorem 10. [18] *If $\alpha_1, \alpha_2, \alpha_3$ are not integers, while $\alpha_4, \ldots, \alpha_n$ are integers, then the necessary and sufficient condition of existence of non-coaxial metric with these angles is (11) with strict inequality. The number of classes of such metrics is at least 1 and at most $\alpha_4 \cdot \ldots \cdot \alpha_n$. The upper bound is attained for generic location of singularities.*

Example. Consider the angles $(1/2, 1/2, 1/2, m)$, where m is an integer. Theorem 10 implies that there are m metrics with these angles for generic location of singularities. These metrics can be lifted on the torus via the 2-to-1 covering ramified over the four singular points. The resulting metric on the torus has one singularity with angle $2m$, and we can apply Theorem 7 to compute the degree in this case. Using the notation introduced before Theorem 7, we obtain

$$g(x) = (1 + x + \ldots)(1 - x^{2m}) = 1 + x + \ldots + x^{2m-1} - x^{2m} + \ldots,$$

so $k = m - 1$ and

$$d = \sum_{0}^{m-1} 1 = m.$$

So in this case, the degree is equal to the number of metrics, and all metrics on the torus come from the sphere via the lifting, so they are invariant with respect to the conformal involution of the torus.

All results in sections 7–9 are all of the same type: once the restriction on the angles is satisfied, a metric exists with any position of singularities, the number of classes of metrics is always finite, and this number is constant for generic singularities. The general reason for this is that the equation on the accessory parameters which gives unitarizable monodromy is algebraic. We conjecture that there are no other cases when the equation for accessory parameters is algebraic.

In the next section we address the only case studied so far when the equation on the accessory parameter is transcendental, and for this case we will see that existence of the metric depends on the location of singularities in an essential way.

10. Angles $(1/2, 1/2, 1/2, 3/2)$ on the sphere, or 3 on a torus.

Let the singularities be a_1, a_2, a_3, a_4. Consider the 2-sheeted ramified covering $\pi : \mathbf{T} \to S$ by a torus with critical values a_j. The metric pulls back to the torus, and we obtain a metric $\rho^*|dz|$ on T with one singularity with angle 3. If we set

$$\rho^*(z) = \frac{1}{\sqrt{2}} e^{u(z)/2},$$

then u will satisfy

$$\Delta u + e^{2u} = 8\pi\delta, \tag{12}$$

where δ is the delta function at 0. Metrics obtained by such pull-back are even, so

$$u(z) = u(-z).$$

Conversely, any even solution of (12) corresponds to a metric on the sphere with 4 conic singularities with angles $(1/2, 1/2, 1/2, 3/2)$. The metric ρ^* is coaxial, so each solution of (12) comes with a one-parametric family of such solutions. This family contains exactly one even metric which corresponds to a metric on the sphere, see [26], [12, Theorem 1].

Equation (12) corresponds to the Lamé equation on the torus $\mathbf{T}$

$$w'' - (2\wp(z) + \lambda)\, w = 0. \tag{13}$$

We denote by $F = w_1/w_2$ a ratio of two linearly independent solutions, this is the developing map of the metric on the torus, and $F = f \circ \pi$, where f is the developing map on the sphere.

The question now becomes: *how many values of accessory parameter λ exist for which the projective monodromy of (13) is unitarizable?*

The answer depends on the parameter $\tau = \omega_1/\omega_2$ of the torus, where we denote the fundamental periods by $2\omega_1, 2\omega_2$. We also set $\omega_3 = \omega_1 + \omega_2$, $e_j = \wp(\omega_j)$, and η_j is defined by $\zeta(z + \omega_j) = \zeta(z) + \eta_j$, where ζ is the Weierstrass ζ-function.

Theorem 11. *For a given τ, there is at most one λ for which (13) has unitarizable projective monodromy. The region in the τ-plane for which such λ exists is explicitly described by the inequalities:*

$$\operatorname{Im}\left(\frac{2\pi i}{e_j\omega_1^2 + \eta_1\omega_1} - \tau\right) < 0, \quad j = 1, 2, 3. \tag{14}$$

The region defined by (14) is shaded in Fig. 1.

Proof. Hermite found an explicit formula for the general solution of (13) see, for example [23, Ch. II, 59]. Let a be a solution of $\wp(a) = \lambda$. Then

$$w_{1,2} = e^{\mp z\zeta(a)}\frac{\sigma(z \pm a)}{\sigma(z)}$$

is a fundamental set of solutions of (13). So their ratio is

$$F(z) = e^{2z\zeta(a)}\frac{\sigma(z - a)}{\sigma(z + a)}.$$

To determine the projective monodromy we compute $F(z + 2\omega)$ where 2ω is a fundamental period:

$$F(z + 2\omega) = e^{4\omega\zeta(a) - 4\eta a}F(z),$$

where we used the formula

$$\sigma(z + 2\omega) = -e^{2\eta(z+\omega)}\sigma(z).$$

So the monodromy is unitarizable if and only if both expressions

$$\omega_1\zeta(a) - \eta_1 a \quad \text{and} \quad \omega_2\zeta(a) - \eta_2 a \quad \text{are pure imaginary.}$$

This means that two equations with respect to a and $\zeta = \zeta(a)$ hold:

$$\omega_1\zeta + \overline{\omega_1}\overline{\zeta} - \eta_1 a - \overline{\eta_1 a} = 0, \tag{15}$$

and

$$\omega_2\zeta + \overline{\omega_2}\overline{\zeta} - \eta_2 z - \overline{\eta_2 a} = 0. \tag{16}$$

Eliminating $\overline{\zeta}$ we obtain one linear equation of the form

$$Aa + B\overline{a} + \zeta(a) = 0, \tag{17}$$

where

$$A = \frac{\pi}{4\omega_1^2 \mathrm{Im}\,\tau} - \frac{\eta_1}{\omega_1}, \quad B = -\frac{\pi}{2|\omega_1|^2 \mathrm{Im}\,\tau}. \tag{18}$$

These constants are uniquely defined by the condition that our equations (15), (16) are invariant with respect to the substitution

$$(a, \zeta) \mapsto (a + 2\omega_k, \zeta + 2\eta_k).$$

Equation (17) must be solved with respect to a. It was proved in [2] that besides the three trivial solutions $a = \omega_k$, $1 \leq k \leq 3$, equation (17) has either none or two solutions of the form $\pm a$. Trivial solutions do not define linearly independent w_1 and w_2 (function F is constant). The two non-trivial solutions $\pm a$ of (17), when exist, define the same $\lambda = \wp(a)$. This proves that at most one such λ exists for any torus. The region D in the space of tori in which such λ exists is described in [2]. The explicit description of D is the following for all $j \in \{1, 2, 3\}$,

$$e_j\omega_1^2 + \eta_1\omega_1 \neq 0 \quad \text{and} \quad \mathrm{Im}\left(\frac{\pi i}{e_j\omega_1^2 + \eta_1\omega_1} - 2\tau\right) < 0.$$

Theorem 11 follows.

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