Metrics of constant positive curvature with conic singularities

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We consider Riemannian metrics of constant curvature κ, with finitely many conic singularities with angles $2\pi\alpha_j$ on a compact surface S of genus g.

WLOG we assume that $\kappa \in \{0, 1, -1\}$. The goal is to understand the space of such metrics for given κ, g and $\alpha = (\alpha_1, \ldots, \alpha_n)$. The topology on the space of metrics is bi-Lipschitz: two metrics ρ_1, ρ_2 are close if there is a homeomorphism $\phi : (S, \rho_1) \to (S, \rho_2)$ such that both ϕ and its inverse have Lipschitz constants close to 1. We denote the space of such metrics by $\mathrm{Met}_{\kappa,g,\alpha}$.

Since each such metric defines a complex structure, we have the *forgetful map* to the moduli space of Riemann surfaces of genus g with n punctures,

$$\mathrm{Met}_{\kappa,g,\alpha} \to \mathrm{Mod}(g, n).$$

The problems we address are the following:

a) What is the topology of $\mathrm{Met}_{\kappa,g,\alpha}$? In particular when it is non-empty, and how many components it has?

b) Is the forgetful map surjective?

c) What the cardinality of the preimage of a point under the forgetful map can be?

These problems can be restated in the language of PDE. Let z be a (flat) conformal local coordinate of S, then a metric can be described by its length element $\rho(z)|dz|$ and constant curvature κ with conic singularities at

a_j means that ρ solves the following PDE

$$\Delta \log \rho + \kappa \rho^2 = 2\pi \sum_{j=1}^{n} (\alpha_j - 1)\delta_{a_j}, \tag{1}$$

or equivalently $\Delta \log \rho + \kappa \rho^2 = 0$ away from the singularities and

$$\rho(z) \sim c_j |z - a_j|^{\alpha_n - 1}, \quad z \to a_j$$

near the singularities.

1. Gauss–Bonnet theorem gives that

$$\chi(S) + \sum_{j=1}^{n} (\alpha_j - 1) = \frac{\kappa}{2\pi} \cdot \text{area}, \tag{2}$$

(each singularity contributes an atom of charge $2\pi(1-\alpha)$ to the curvature), so one necessary condition for existence of the metric is that the LHS of (2) has the same sign as κ.

When $\kappa \leq 0$, there are complete answers to all our questions:

$\mathrm{Met}_{\kappa,g,\alpha} \neq \emptyset$ if and only if the Gauss-Bonnet restriction is satisfied, that is the LHS of (2) is negative when $\kappa = -1$ or zero when $\kappa = 0$. If this is satisfied with $\kappa = -1$ the metric is unique, and for $\kappa = 0$ it is unique up to a multiplicative constant.

This result is due to Picard, who wrote 4 long papers on it in 1893–1931. Two different modern proofs were given by M. Heins in 1962 and M. Troyanov in 1991.

When $\chi(S) < 0$, $\kappa = -1$ and $n = 0$, Picard's result is equivalent to the uniformization theorem, and when $\kappa = 0$ to the Schwarz–Christoffel formula.

So from now on we discuss only the case $\kappa = 1$, and denote $\mathrm{Met}_{1,g,\alpha}$ by $\mathrm{Sph}_{g,\alpha}$.

If $g = 0$ and all $\alpha_j \in (0,1)$ we also have a complete result: the necessary and sufficient condition for existence of the metric is

$$0 < 2 + \sum_{j=1}^{n} (\alpha_j - 1) \leq 2 \min_j \alpha_j \tag{3}$$

and when it is satisfied, the metric is unique. This is due to Feng Luo and Gang Tian (1992). Notice the new condition on the angles in the RHS.

2. Developing map. Since a surface of curvature 1 is locally isometric to a region on the standard unit sphere, an analytic continuation of this isometry gives a multivalued holomorphic *developing map*

$$f : S\backslash A \to \overline{\mathbf{C}},$$

where $A = \{a_1, \ldots, a_n\}$ is the set of singularities. The metric is recovered from the developing map by the formula

$$\rho(z) = \frac{2|f'(z)|}{1 + |f(z)|^2},$$

and the developing map is characterized by two properties: first in appropriate coordinates on $\overline{\mathbf{C}}$ we have

$$f(z) \sim c(z - a_j)^{\alpha_j}, \quad z \to a_j, \quad 1 \leq j \leq n,$$

and the second property is that the monodromy of f consists of rotations: this means that the result f_γ of an analytic continuation of f along a path γ satisfies

$$f_\gamma = \phi_\gamma \circ f, \quad \phi_\gamma \in PSU(2) \approx SO(3).$$

These two properties characterize all possible developing maps. Two such developing maps f_1, f_2 define the same metric iff $f_1 = \phi \circ f_2$, for some $\phi \in PSU(2)$.

The image of the monodromy representation $\gamma \mapsto \phi_\gamma$ is called the monodromy group. It is defined up to conjugation in $PSU(2)$.

It may happen that two developing maps f_1, f_2 corresponding to different spherical metrics are related by

$$f_1 = \phi \circ f_2, \quad \text{where} \quad \phi \in PSL(2)\backslash PSU(2). \tag{4}$$

One can show that this is only possible when the monodromy group is conjugate to a subgroup of rotations of a circle, and we call such metrics and their developing maps *co-axial*. Two metrics will be called *equivalent* if they satisfy (4).

General conjecture. *To every conformal structure correspond finitely many equivalence classes of metrics.*

This conjecture is proved in E 2020 in the following cases: $(g, n) = (1, 1)$ and $(g, n) = (0, n)$, $n \leq 4$ that is exactly in those cases when the moduli space and the space of metrics is one-dimensional.

3. Angle restrictions for spherical metrics:

$$\chi(S) + \sum_{j=1}^{n} (\alpha_j - 1) > 0 \tag{5}$$

and

$$\text{When} \quad g = 0, \quad d_1(\alpha - 1, \mathbf{Z}_o^n) \geq 1. \tag{6}$$

Here d_1 is ℓ^1 distance in $\mathbf{R}^n$,

$$\alpha - 1 = (\alpha_1 - 1, \ldots, \alpha_n - 1),$$

and $\mathbf{Z}_o^n$ is the part of the integer lattice consisting of those points whose sum of coordinates is odd.

Necessity of condition (6) is due to G. Mondello and D. Panov, 2016; it generalizes various special cases obtained earlier, for example the RHS inequality in (3) is a consequence of it.

The same authors showed that (5) and *strict inequality* in (6) are sufficient for the existence of a metric, and all metrics on the sphere which satisfy (5) with equality are co-axial.

Possible angles of co-axial metrics for $g = 0$ were completely described in E 2020, see Appendix A, so we know exactly under what conditions $\mathrm{Sph}_{g,\alpha} \neq \emptyset$.

When $g \geq 1$, the necessary and sufficient condition on the angles for existence of a metric is (5), and the metric can be co-axial only when $g = 1$ and all α_j are integers.

4. Topology of $\mathrm{Sph}_{g,\alpha}$**.** In MP 2019, an example of disconnected space $\mathrm{Sph}_{g,\alpha}$ is given. The complete description of topology of such a space is only known when $(g, n) = (1, 1)$. To state it we recall the notion of 2-dimensional orbifold. It is a compact surface S equipped with a function $n : S \to \mathbf{Z}_{\geq 1} \cup \{\infty\}$ which takes values > 1 only at finitely many points. The orbifold Euler characteristic χ^O is defined as

$$\chi^O = \chi(S) - \sum_S \left(1 - \frac{1}{n(x)}\right).$$

The points with $n(x) \geq 2$ are called *orbifold points of order* n, and the points with $n(x) = \infty$ correspond to punctures.

Theorem. *Denote* $m = [(\alpha+1)/2]$. *When* α *is not an odd integer,* $\mathrm{Sph}_{1,\alpha}$ *is a connected orientable surface of genus* $[(m^2-6m+12)/12]$ *with* m *punctures.*

It has a natural orbifold structure with one orbifold point of order 3 *iff* $d_1(\alpha, 6\mathbf{Z}) > 1$ *and one orbifold point of order 2 iff* $d_1(\alpha, 2\mathbf{Z}) > 1$, *and no other orbifold points. The orbifold Euler characteristic is*

$$\chi^O = -m^2/6.$$

When $\alpha = 2m$ is an even integer, there is a natural complex analytic structure on $\mathrm{Sph}_{1,2m}$ which turns it into a Belyi curve, and the forgetful map is is holomorphic and even algebraic,

Theorem. $\mathrm{Sph}_{1,2m+1}$ *is a manifold of dimension* 3 *with* $[m(m+1)/6]$ *components. Each component is* $D \times R$, *where* D *is an open disk. The metrics are co-axial. The set of equivalence classes of metrics consists of* $[m(m+1)/6]$ *disks. There is a natural orbifold structure: when* $m \equiv 1 \pmod 3$, *one of the disks has an orbifold point of order* 3, *and there are no other orbifold points.*

5. Generic angles. For positive numbers $\alpha_1, \ldots, \alpha_n$ we define the set

$$\mathrm{Crit}_{g,\alpha} := \{\|\alpha_I\| - \|\alpha_{cI}\| + 2b : I \subset \{1, \ldots, n\}, b \in \mathbf{Z}_{\geq 0}\},$$

where

$$\|\alpha_I\| := \sum_I \alpha_j, \quad cI = \{1, \ldots, n\} \backslash I.$$

Then we define the number

$$NB_{g,\alpha} = d_{\mathbf{R}}\left(\chi(S \backslash A), \mathrm{Crit}_{g,\alpha}\right),$$

where $A = \{a_1, \ldots, a_k\} \subset S$ and $d_{\mathbf{R}}$ is the usual distance on the real line.

This number $NB_{g,\alpha}$ is called the *non-bubbling parameter.* For example, it is positive when there are no integer relations of the form

$$\sum_I \alpha_j - \sum_{cI} \alpha_j = \text{integer}.$$

Theorem. (MP 2019) *If* $NB_{g,\alpha} > 0$ *then the forgetful map is proper.*

It is possible that $NB_{g,\alpha} > 0$, and the forgetful map is not surjective (MP 2019, Theorem D.) However we have

Theorem (Bartolucci, de Marchis, Malchidi, 2011) *If* $g \geq 1$, *and* $NB_{g,\alpha} > 0$, *and all* $\alpha_j > 1$, *and*

$$0 < 2 + \sum_j (\alpha_j - 1) > 2 \min\{\alpha_1, \ldots, \alpha_n, 1\},$$

then the forgetful map is surjective.

C.-C. Chen and C.-S. Lin (2015) computed the Leray-Schauder degree of the non-linear operator in (1) with $\kappa = 1$, which also gives the Corollary for generic α and arbitrary g. Their formula for the degree is somewhat complicated, and it is given in Appendix B.

6. Special cases.

a). The most interesting of known special cases where all questions are answered is $(g, n) = (1, 1)$, $\alpha = 3$. It is equivalent to the case $(g, n) = (0, 4)$, $\alpha = (1/2, 1/2, 1/2, 3/2)$.

C.-S. Lin and C.-L. Wang proved in this case that the forgetful map is injective but not surjective, and explicitly described its image. Their proof is very complicated, and a simpler proof was obtained by Bergweiler and Eremenko in 2016. The image is an unbounded region in $\mathbf{C} = \mathrm{Mod}_{1,1}$ whose boundary is an analytic curve described by an explicit equation.

This was partially generalized in EMP 2020: for $\mathrm{Sph}_{1,2m+1}$ the forgetful map is not proper, and the boundary of its image in $\mathrm{Mod}(1, 1)$ is explicitly described.

b). When $g = 0$, and at most 3 of the α_j are not integers, the forgetful map is complex-analytic (with respect to some natural complex-analytic structure on $\mathrm{Met}_{0,\alpha}$), so degree of this map is equal to the number of preimages of a generic point. All these cases have been studied in detail by Eremenko, Gabrielov and Tarasov. We state the result for 3 generic non-integer angles (the rest of the angles are integers).

Suppose that $\alpha_1, \alpha_2, \alpha_3$ are not integers, and the rest $\alpha_4, \ldots, \alpha_n$ are integers, and no alternating sum

$$\alpha_1 \pm \alpha_2 \pm \alpha_3 \quad \text{is an integer.}$$

Then the forgetful map is holomorphic (in fact algebraic and finite), and its degree is the product $\alpha_4 \ldots \alpha_n$. In particular, the generic point has $\alpha_4 \ldots, \alpha_n$ distinct preimages.

c). An interesting special case is when $g = 0$ and all α_n are integers. In this case the developing map is a rational function, and the forgetful map is the co-called Wronski map. When all $\alpha_j = 2$, the degree of the forgetful map is the Catalan number (L. Goldberg, 1991), in the general case the formula is more complicated (I. Scherbak, 2001, EGSV, 2006).

7. Some unsolved questions.

Is the forgetful map always open? Is it open when $(g, n) = (1, 1)$?

How to estimate the number of preimages under the forgetful map from above? Even in the simplest case of one-dimensional spaces when the preimage is known to be finite (see section 2), no upper estimate is known.

What is the topology of $\mathrm{Sph}_{0,\alpha}$ when $n = 4$?

Appendix A. Condition on the angles for co-axial metrics on the sphere.

Let us write $\alpha = (\alpha_1, \ldots, \alpha_n)$ so that α_j are not integers for $j \leq m$, while $\alpha_{m+1}, \ldots, \alpha_n$ are integers.

Theorem. *For the existence of a spherical co-axial metric on the sphere with angles α, it is necessary that:*

(i) there exists a choice of signs $\epsilon_j \in \{\pm 1\}$ and a positive integer k' such that

$$\sum_{j=1}^{m} \epsilon_j \alpha_n = k', \tag{7}$$

and

(ii) the integer

$$k'' := \sum_{j=m+1}^{n} \alpha_j - n - k' + 2 \quad \textit{is even and non-negative.}$$

If the numbers c_j in

$$\mathbf{c} := (c_1, \ldots, c_q) := (\alpha_1, \ldots, \alpha_m, \underbrace{1, \ldots, 1}_{k'+k''\ \textit{times}})$$

are incommensurable, then (i) and (ii) are also sufficient.

(iii) If $\mathbf{c} = \eta\mathbf{b}, \eta \neq 0$ *and the coordinates of* $\mathbf{b}$ *are integers with whose greatest common factor is* 1*, then there is an additional necessary condition*

$$2 \max_{m+1 \leq j \leq n} \alpha_j \leq \sum_{j=1}^{q} b_j, \tag{8}$$

and in this case the three conditions (i), (ii) and (8) are sufficient.

Appendix B. Leray–Shauder degree of (3) according to Chen and Lin. Consider the following generating function

$$g(x) = (1 + x + x^2 + \ldots)^{n-\chi(S)} \prod_{j=1}^{n} (1 - x^{\alpha_k}).$$

Let the the power series expansion at 0 be

$$g(x) = 1 + b_1 x^{n_1} + b_2 x^{n_2} + \ldots + b_k x^{n_k} + \ldots.$$

Theorem. *Suppose that* $NB_{g,\alpha} > 0$*. Then there exists a unique integer* k *satisfying*

$$2n_k < \chi(S) + \sum_{j=1}^{n} (\alpha_j - 1) < 2n_{k+1},$$

and the Leray–Schauder degree of the operator in the LHS of (3) is

$$d = \sum_{j=0}^{k} b_j.$$

References

[1] D. Bartolucci F., de Marchis and A. Malchiodi, Supercritical conformal metrics on surfaces with conical singularities, Int. Math. Res. Not., no. 24 (2011) 5625—5643.

[2] W. Bergweiler and A. Eremenko, Green functions and antiholomorphic dynamics on tori, Proc. AMS, 144, 7 (2016) 2911—2922.

[3] Chiun-Chuan Chen and Chang-Shou Lin, Mean field equation of Liouville type with singular data: topological degree, Comm. Pure Appl. Math., 68 (2015) 887—947.

[4] A. Eremenko, Co-axial monodromy, Ann. Sc. Norm. Super. Pisa, Cl. di Sci. (5) 20, 2 (2020) 619–634. arXiv:1706.04608.

[5] A. Eremenko, Metrics of constant positive curvature with four conic singularities on the sphere, Proc. AMS 148, 9 (2020) 3957–3965. arXiv:1905.02537.

[6] A. Eremenko, G. Mondello and D. Panov, Moduli of spherical tori with one conical point, arXiv:arXiv:2008.02772.

[7] A. Eremenko and V. Tarasov, Fuchsian equations with three nonapparent singularities, SIGMA, 14 (2018), 058, 12 pages, www.emis.de/journals/SIGMA/2018/058/.

[8] M. Heins, On a class of conformal metrics, Nagoya Math. J., 21 (1962) 1–60.

[9] C-S. Lin and Wang, Elliptic functions, Green functions and the mean field equations on tori, Ann. of Math. (2) 172 (2010), no. 2, 911– 954.

[10] Feng Luo and Gang Tian, Liouville equation and spherical convex polytopes, Proc. Amer. Math. Soc. 116 (1992), no. 4, 1119–1129.

[11] G. Mondello and D. Panov, Spherical metrics with conical singularities on a 2-sphere: angle constraints. Int. Math. Res. Not. IMRN 2016, no. 16, 4937–4995.

[12] G. Mondello and D. Panov, 2019 Spherical metrics with conical points: systole inequality and moduli spaces with many connected components, GAFA 29 (2019) 1110-–1193. arXiv:1807.04373.

[13] E. Picard, De l'équation $\Delta u = ke^u$ sur une surface de Riemann fermée, J. Math. Pures Appl 9 (1893) 273—292.

[14] E. Picard, Quelques applications analytiques de la théorie des courbes et des surfaces algébriques, Gauthier-Villars, Paris, 1931.

[15] M. Troyanov, Prescribing curvature on compact surfaces with conical singularities, Trans. Amer. Math. Soc., 324 (1991) 793–821.