Gluing linear differential equations

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In this talk I describe the recent work joint with Walter Bergweiler where we introduce a method of construction of linear differential equations such that the asymptotic distribution of zeros of their solutions in the complex plane can be controlled.

On the Bank-Laine conjecture, arXiv:1408.2400,
Quasiconformal surgery and linear differential equations, arXiv:1510.05731.

We consider ordinary differential equations

$$w'' + Aw = 0, \tag{1}$$

where A is an entire function. All solutions are entire functions and we are interested in the distribution of zeros of solutions in the complex plane. The order of an entire function f is defined by

$$\rho(f) = \limsup_{z\to\infty} \frac{\log\log|f(z)|}{\log|z|},$$

and the exponent of convergence of zeros z_k as

$$\lambda(f) = \inf\{\lambda > 0 : \sum_{k:z_k\neq 0} |z_k|^{-\lambda} < \infty\}.$$

Inequality $\lambda(f) \leq \rho(f)$ always holds; it follows from Jensen's theorem.

When A is a polynomial, a very precise asymptotic theory is available; it gives that the order of every solution is $(d+2)/2$, where $d = \deg(A)$. Some solutions can be free of zeros, but if $E = w_1 w_2$ is the product of two linearly independent solutions, then we have $\lambda(E) = (d+2)/2$, unless A is a constant.

Some equations with transcendental entire coefficient A were intensively studied since 19-th century, like the Mathieu and Hill's equations, but the general theory of such equations begins with the work of Bank and Laine in the early 1980-th. The following facts were established.

1. *There can be two linearly independent solutions without zeros, for example*

$$w'' - (1/4)(e^{2p} + (p')^2 - 2p'')w = 0, \tag{2}$$

where p *is a polynomial, has solutions*

$$w_{1,2}(z) = \exp\left(-\frac{1}{2}\left(p(z) \pm \int_0^z e^{p(\zeta)} d\zeta\right)\right).$$

However this is only possible only when $\rho(A)$ *is a positive integer or infinity.*

2. *If* A *is transcendental,* $\rho(A)$ *is finite and not a positive integer, then*

$$\lambda(E) \geq \rho(A). \tag{3}$$

Moreover, if

$$\rho(A) \leq 1/2 \quad \textit{then} \quad \lambda(E) = \infty. \tag{4}$$

and

$$\frac{1}{\lambda(E)} + \frac{1}{\rho(A)} \leq 2, \tag{5}$$

when $\rho(A) \in (1/2, 1)$.

Based on these results, Bank and Laine conjectured that if A is transcendental, and $\rho(A)$ is not a positive integer, then $\lambda(E) = \infty$.

This conjecture raised a considerable interest, and there are many results where conditions on A are imposed which imply that $\lambda(E) = \infty$. We give two such results. One, due to Bank, Laine and Langley, states conditions in terms of behaviour of $|A|$ on rays:

3. *Suppose that for almost every* θ *we have either*

$$r^{-N}|A(re^{i\theta})| \to \infty, \quad r \to \infty \quad \textit{for all} \quad N > 0, \tag{6}$$

or

$$|A(re^{i\theta})| = O(r^n), \quad r \to \infty, \quad \textit{where} \quad 2 < n + 2 < 2\rho(A), \tag{7}$$

or

$$\int_0^\infty r|A(re^{i\theta})|dr < \infty. \tag{8}$$

Then $\lambda(E) = \infty$.

Condition on $n + 2 < 2\rho(A)$ in (7) is best possible as the example (2) shows. Condition (8) was added to cover the case $\rho(A) < 1$ when the condition on n in (7) is not applicable.

Another result, due to Toda, improves (5) when an additional information on A is available:

4. *If* $\rho(A) < \infty$ *and the set* $\{z : |A(z)| > c\}$ *has* N *components for some* $c > 0$ *then* $\rho(A) \geq N/2$, *and*

$$\frac{1}{\lambda(E)} + \frac{N}{\rho(A)} \leq 2,$$

so $\rho(A) = N/2$ *implies that* $\lambda(E) = \infty$.

One can slightly improve these results using the same methods which their authors used:

3a. *Suppose that* $\rho(A) \in (1/2, 1)$, *and for almost every* θ *either (6) or (7) holds with* $0 < n < 2\rho(A)/(4\rho(A) - 2)$. *Then* $\lambda(E) = \infty$.

4a. *If* $\rho(A) < \infty$ *and the set* $\{z : |A(z)| > c\}$ *has* N *components, then we have* $\rho(A) \geq N/2$ *and*

$$\frac{N}{\lambda(E)} + \frac{N}{\rho(A)} \leq 2.$$

All results mentioned so far are based on consideration of the following non-linear differential equation satisfied by E:

$$-2\frac{E''}{E} + \left(\frac{E'}{E}\right)^2 - \frac{W}{E^2} = 4A. \tag{9}$$

Here $W = w_1 w_2' - w_1' w_2$ is the constant Wronskian of two linearly independent solutions of (1). Equation (9) shows that in the places where E is large, A cannot be very large, and this leads to conclusions of 2-4a.

The general solution of (9) with entire A is an entire function, which is the product of two linearly independent solutions of (1) with Wronskian W.

When $W=1$, such entire functions E can be characterized in the following way: *whenever* $E(z)=0$ *we have* $E'(z)\in\{\pm 1\}$. Entire functions with this property are called the *Bank–Laine functions.* They were much studied since the early 1980-th, but the following basic question remained unsolved: what are the possible orders of Bank–Laine functions? It follows from 2, that the order must be at least 1. There are elementary examples where the order is an integer or infinity. Langley showed that there are many Bank–Laine functions of finite order, besides these elementary examples, however in all cases when the order could be determined, it was an integer.

Our contribution which is described below is a construction of Bank–Laine functions of arbitrary order ≥ 1. Moreover, our constructions show that all results 2, 3a, 4a are best possible.

Let $F=w_2/w_1$ be the ratio of two linearly independent solutions of (1). This is a *locally univalent* meromorphic function which satisfies the Schwarz differential equation

$$S(F):=\frac{F'''}{F'}-\frac{3}{2}\left(\frac{F''}{F}\right)^2=2A.$$

There is a bijective correspondence $F\mapsto S(F)=2A$ between all locally univalent meromorphic functions F modulo post-composition with fractional linear transformations, and all entire functions A.

The Schwarzian admits a factorization: $2S(F)=B(F/F')$, where

$$B(E)=-2\frac{E''}{E}+\left(\frac{E'}{E}\right)^2-\frac{1}{E^2},$$

is the Bank–Laine operator. If $F=w_2/w_1$ then $F/F'=w_1w_2/W$, where W is the Wronskian of w_1 and w_2.

In our constructions, the principal object is a meromorphic locally univalent function F. Consider the following example:

$$F(z)=P(e^z)e^{e^z},$$

where P is a polynomial. The local univalence condition implies that

$$F'(z)=(P'(e^z)+P(e^z))e^ze^{e^z}\neq 0,$$

so $P' + P$ must be a monomial. This implies that $P(w)$ is a partial sum of the Taylor series of e^{-w}. For a technical reason, we take a partial sum of even degree:

$$P_m(w) = \sum_{k=0}^{2m} \frac{(-w)^k}{k!}.$$

To this choice of F correspond $E(z) = (2m)!e^{(2m-1)z}P_m(e^z)$ and

$$A(z) = -(1/4)e^{2z} - me^z - (2m+1)^2/4,$$

both are entire functions of order 1.

To obtain examples with different asymptotic behavior, we glue several such functions together. Let

$$g_m(z) = P_m(e^z)e^{e^z}.$$

These are $2\pi i$-periodic entire functions, which map the real line onto $(1, +\infty)$ homeomorphically. Our choice of degree $2m$ of P_m ensures that this map is increasing.

Example 0.

The simplest examples are obtained when we glue together two such functions, g_m restricted to the upper half-plane H^+, and g_n restricted to the lower half-plane H^-. Let ϕ be the increasing homeomorphism of the real line defined by

$$g_m(x) = g_n(\phi(x)), \quad x \in \mathbf{R}.$$

This homeomorphism has the following asymptotic behavior:

$$\phi(x) = \begin{cases} x + O(e^{-\delta z}), & x \to +\infty \\ kx + b + O(e^{-\delta|x|}), & x \to -\infty. \end{cases}, \tag{10}$$

where $k = (2m+1)/(2n+1)$.

We want to find a curve γ which divides the plane into two parts D^+ and D^- and quasiconformal homeomorphisms $h^\pm : D^\pm \to H^\pm$ such that $h^-(z) = \phi(h^+(z)),\ z \in \gamma$. Then the function

$$G(z) = \begin{cases} g_m(h^+(z)), & z \in D^+, \\ g_n(h^-(z)), & z \in D^- \end{cases}$$

will be continuous in the plane:

$$g_m(h^+(z)) = g_n(\phi(h^+(z)) = g_n(h^-(z))), \quad z \in \gamma,$$

see Fig. 1[1], and hopefully G will be quasiconformal with small dilatation. In fact there exist conformal maps $h^\pm$ which do the job, but we want to know explicitly their asymptotic behaviour, and to do this it is convenient to work instead with quasiconformal maps. Small dilatation means

$$\int_{\mathbf{C}} \frac{K_G(x,y)-1}{x^2+y^2} dxdy < \infty, \tag{11}$$

where $K_G \geq 1$ is the quasiconformal dilatation. This condition implies according to Teichmüller, Wittich and Belinskii, that there exists a quasiconformal homeomorphism ψ of the plane such that $F = G \circ \psi$ is an entire function, and

$$\psi(z) \sim z, \quad z \to \infty. \tag{12}$$

This function F is locally univalent by construction, and its asymptotic behaviour can be determined if we know the asymptotic behavior of $h^\pm$, because g_m, g_n are explicitly known and ψ is sufficiently controlled by (12).

To construct the gluing $\gamma, h^\pm$, let us first consider the homeomorphism ϕ^* which is obtained by neglecting the small error terms in (10):

$$\phi^*(x) = \begin{cases} x, & x > 0, \\ kx, & x < 0. \end{cases} \tag{13}$$

This is elementary, and $(h^*)^\pm$ can be chosen conformal. Indeed, the principal branch of $q(z) = z^\mu$ with an appropriate complex μ maps the plane with a cut along the negative ray onto the complement of the logarithmic spiral, such that

$$q(x+0i) = q(kx-0i), \quad x < 0.$$

See Fig. 2. The image of the positive ray under q is another logarithmic spiral, and the union of these two spirals is a curve γ which divides the plane into D^+ and D^-. Our homeomorphisms $(h^*)^\pm : D^\pm \to H^\pm$ are the branches of q^{-1}. A simple computation gives

$$\mu = \frac{2\pi}{4\pi^2 + \log^2 k}(2\pi - i \log k).$$

[1]Figures are in the end of the paper.

To perform a similar gluing with our actual homeomorphism ϕ, one has to modify these $(h^*)^{\pm}$ by composing them with quasiconformal homeomorphisms which satisfy (12), and thus do not affect the asymptotic behaviour.

Asymptotic behaviour of A and E can now be found because our function F is represented as a composition of explicit functions $g_m, g_n, (h^*)^{\pm}$ and quasiconformal maps that satisfy (12).

This construction gives Bank-Laine functions and potentials A with

$$\rho(E) = \lambda(E) = \rho(A) = 1 + \frac{\log^2 k}{4\pi^2}, \quad k = \frac{2m+1}{2n+1},$$

so we achieve a dense set of orders in $(1, +\infty)$.

To achieve all orders in $(1, +\infty)$ and to construct further examples, we need to glue together an infinite set of functions g_m.

Asymptotic behaviour of $\phi_{m,n}$ when $m, n \to \infty$.

To control the dilatation of gluing of infinitely many g_m we need asymptotic behavior of homeomorphisms $\phi_{m,n}$ defined by

$$g_m(x) = g_n(\phi_{m,n}(x)), \quad x \in \mathbf{R}.$$

This is studied using a version of Szego's asymptotics for partial sums of the exponential:

$$\log(g_m(x) - 1) = -\log k! + e^x + kx - \log\left(1 + \frac{e^x}{k}\right) + R(e^x, k),$$

where $k = 2m + 1$ and

$$|R(w, k)| \leq \frac{24w}{k(k+w)}, \quad w > 0, \quad k \geq 24.$$

Then a computation shows that $|\phi_{m,n}(x) - x|$ and $|\log \phi'_{m,n}(x)|$ are uniformly bounded from above when $Cn > m > n \to \infty$, where C is a positive constant. This, together with a quantitative improvement of (10) is sufficient to verify (11) in our subsequent constructions.

Example 1.

Let us choose a sequence of non-negative integers (m_j), $j \geq 1$, and restrict g_{m_j} to the strips

$$\Pi_j = \{x + iy : 2\pi(j-1) < |y| < 2\pi j\}.$$

We want to glue all these functions together. We have homeomorphisms ϕ_j of the real line defined by

$$g_{m_{j+1}}(x + 2\pi ij) = g_{m_j}(\phi_j(x) + 2\pi ij).$$

where the homeomorphisms behave at infinity similarly to (10) with constants

$$k_j = \frac{2m_{j+1} + 2}{2m_j + 2}.$$

As in (10) above we have

$$\phi_j(x) \approx \begin{cases} x, & x > 0, \\ k_j x, & x < 0. \end{cases} \tag{14}$$

Assuming exact equality in (14) the gluing can be performed in the following way. We define affine maps h_j^{-1} which send the half-strips

$$\Pi_j^- \{x + iy : x < 0,\ 2\pi(j-1) < |y| < 2\pi j\}$$

onto half-strips S_j which fill the left half-plane (see Fig. 3), and such that $g_{m_j} \circ h_j \approx \text{id}$ on $\partial S_j \cap \{z : \Re z < 0\}$, so that the function defined by $g_j \circ h_j$ in S_j is almost continuous in the left half-plane. By doing this we introduced a discontinuity along the imaginary line. This discontinuity is described by a piecewise-linear homeomorphism q, and we choose our sequence m_j so that q has the required asymptotic behaviour. Then we glue the right and left half-plane along q, using power functions.

Choosing

$$q(iy) \approx \begin{cases} iky, & y > 0, \\ iy, & y < 0, \end{cases}$$

where k is a real number, we obtain a function with spiraling behaviour of the type we constructed above, but this time we can achieve any order $\rho(A) = \lambda(E) \in [1, \infty)$. This extends Example 0, and solves the question on the possible orders of Bank–Laine functions.

Example 2.

Choosing

$$q(iy) \approx iy|y|^{\gamma - 1},$$

with some $\gamma > 1$ we obtain another gluing problem which is solved in the first approximation by two power functions, one in the left half-plane, another in the right half-plane (Fig. 4). If $\gamma = 1/(2\rho - 1)$, $\rho \in (1/2, 1)$, these functions are:

$$\Phi + (z) = z^{1/\rho}, \quad |\arg z| < \pi/2,$$

and

$$\Phi^-(z) = -(-z)^{2-1/\rho}, \quad |\arg z - \pi| < \pi/2.$$

This choice produces the following asymptotic behaviour:

$$\log|A(re^{i\theta})| \sim 2\log\frac{1}{|E(re^{i\theta}|} \sim r^\rho \cos\rho\theta, \quad |\theta| < \pi/(2\rho), \tag{15}$$

$$\log|E(-re^{i\theta})| \sim r^\sigma \cos\sigma\theta, \quad |\theta| < \pi/(2\sigma), \tag{16}$$
$$|A(-re^{i\theta})| \sim cr^{2\sigma-2}, \quad |\theta| < \pi/(2\sigma),$$

where $\rho = \rho(A) \in (1/2, 1)$ is prescribed, and

$$\frac{1}{\rho} + \frac{1}{\sigma} = 2.$$

This example shows that (5) and 3a are best possible.

Example 3.

To show that 4a is best possible, we need a function F whose asymptotic behavior is similar to $F_1(z^N)$, where F_1 is the locally univalent function which produces (15), (16). Of course $F_1(z^N)$ is not locally univalent. To construct the required F, we consider F_1 which maps $\mathbf{R}$ homeomorphically onto $(1, +\infty)$ and prepare another locally univalent function F_2 whose asymptotic behavior is similar to F_1 but mapping $\mathbf{R}$ homeomorphically onto $(0, 1)$, changing the orientation. Then we restrict F_1 and F_2 to the upper and lower half-planes, and consider the regions B_j, $1 \leq j \leq 2N$ shown in Fig. 5. These regions are contained in the sectors

$$\Sigma_j = \{z : 2\pi(j-1)/(2N) < \arg z < 2\pi j/(2N)\}$$

and are asymptotically close to these sectors. Let φ_j be conformal maps from B_j to the upper or lower half-plane, and define $G_j = F_1 \circ \phi_j$ for $2 \leq j \leq 2N - 1$ and $G_j = F_2 \circ \phi_j$ for $j = 1, 2N$. Then G maps components of ∂B_j homeomorphically onto $(1, \infty)$ or onto $(0, 1)$ as shown in the figure. Then we map the complementary region B to $\overline{\mathbf{C}}$ by a local homeomorphism whose boundary values match those of G. Choosing the appropriate shape of B we can assure that the resulting quasiconformal map has dilatation satisfying (11).

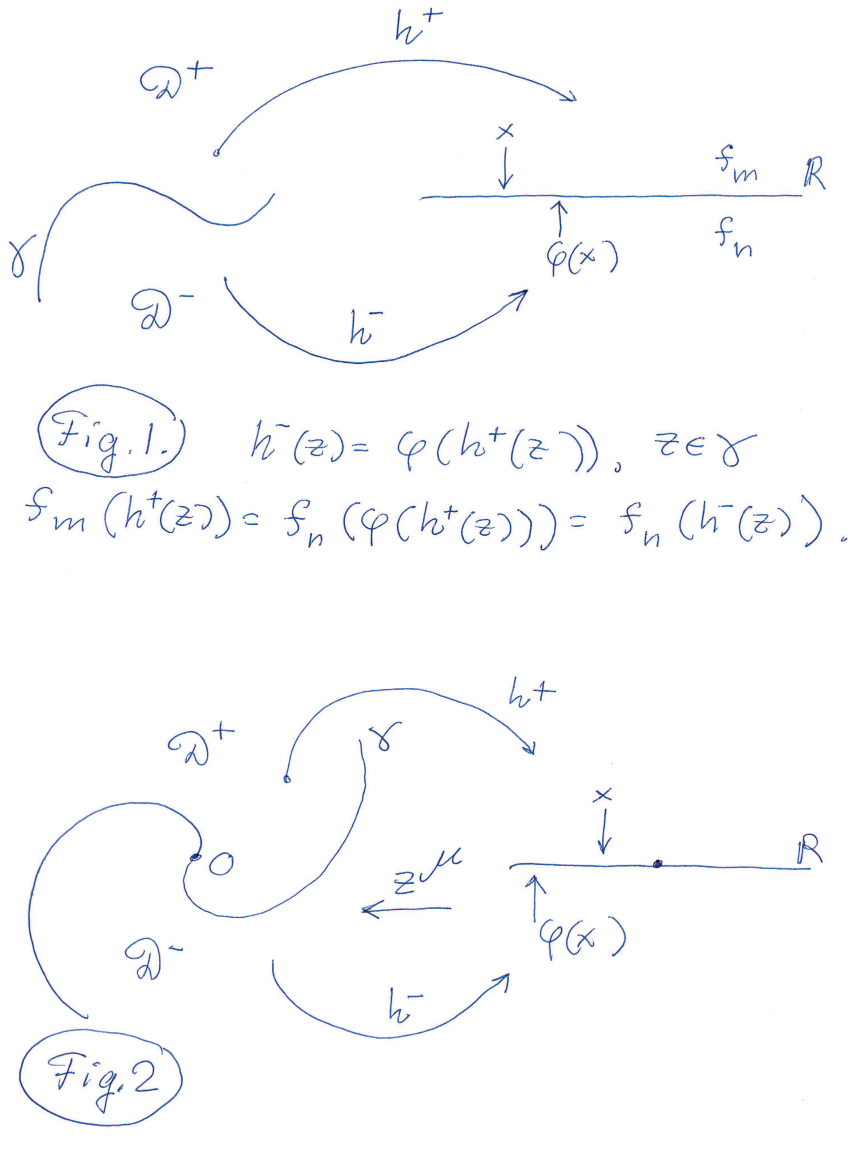

Figure 1: Figures 1 and 2

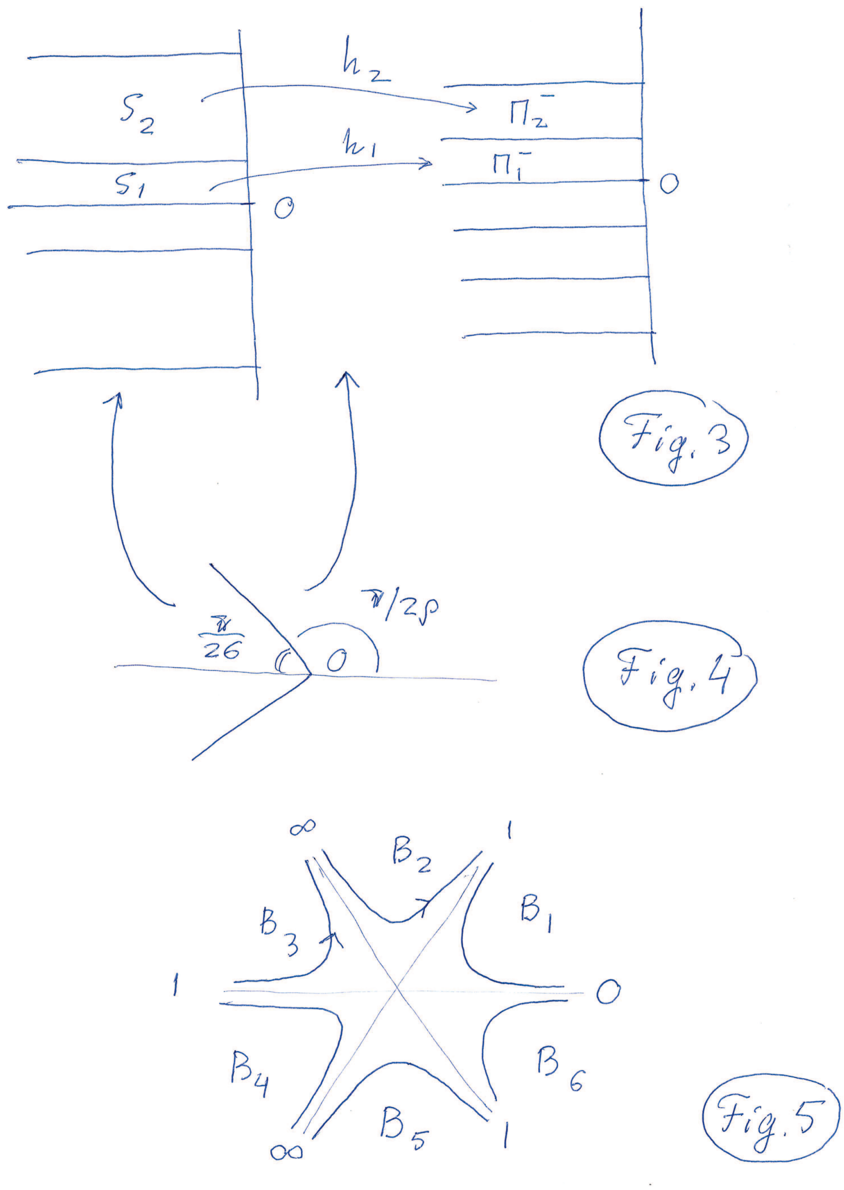

Figure 2: Figures 3-5.