Quasi-exactly solvable quartic: real QES locus

Alexandre Eremenko and Andrei Gabrielov*

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Abstract

We describe the real quasi-exactly solvable locus of the PT-symmetric quartic using Nevanlinna parametrization.

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Following Bender and Boettcher [3] we consider the eigenvalue problem in the complex plane

$$w'' + (\zeta^4 + 2b\zeta^2 + 2iJ\zeta + \lambda)w = 0, \quad w(te^{-\pi i/2 \pm \pi i/3}) \to 0, \quad t \to +\infty, \quad (1)$$

where J is a positive integer. This problem is quasi-exactly solvable [3]: there exist J elementary eigenfunctions $w = p_n \exp(-i\zeta^3/3 - ib)$, where p_n is a polynomial of degree $n = J - 1$.

When b is real, the problem is PT-symmetric. By the change of the independent variable $z = i\zeta$ the problem is equivalent to

$$-y'' + (z^4 - 2bz^2 + 2Jz)y = \lambda y, \quad y(te^{\pm \pi i/3}) \to 0, \quad t \to +\infty. \quad (2)$$

Polynomial h in the exponent of an elementary eigenfunction $y(z)$ is $h(z) = z^3/3 - bz$. The *spectral locus* Z_J is defined as

$$\{(b, \lambda) \in \mathbf{C}^2 : \exists\, y \neq 0 \text{ satisfying (2)}\}.$$

The *real spectral locus* $Z_J(\mathbf{R})$ is $Z_J \cap \mathbf{R}^2$. The *quasi-exactly solvable spectral locus* Z_J^{QES} is the set of all $(b, \lambda) \in Z_J$ for which there exists an elementary

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solution y of (2). This is a smooth irreducible algebraic curve in $\mathbf{C}^2$, [1, 2]. In this paper we describe $Z_J^{QES}(\mathbf{R}) = Z_J^{QES} \cap \mathbf{R}^2$. We prove a result announced in [6]:

Theorem 1. For $n \geq 0$, $Z_{n+1}^{QES}(\mathbf{R})$ *consists of* $[n/2]+1$ *disjoint analytic curves* $\Gamma_{n,m}$, $0 \leq m \leq [n/2]$ *(analytic embeddings of* $\mathbf{R}$ *to* $\mathbf{R}^2$*).*

For $(b,\lambda) \in \Gamma_{n,m}$*, the eigenfunction has* n *zeros,* $n-2m$ *of them real.*

If n *is odd, then* $b \to +\infty$ *on both ends of each curve* $\Gamma_{n,m}$*. If* n *is even, then the same holds for* $m < n/2$*, but on the ends of* $\Gamma_{n,n/2}$ *we have* $b \to \pm\infty$*.*

If $(b,\lambda) \in \Gamma_{n,m}$, $(b,\mu) \in \Gamma_{n,m+1}$ *and* b *is sufficiently large, then* $\mu > \lambda$*.*

This Theorem establishes the main features of the $Z_{n+1}^{QES}(\mathbf{R})$ which can be seen in the computer-generated figure in [3]. Similar results were proven in [5] for two other PT-symmetric eigenvalue problems.

Our Theorem parametrizes *all* polynomials P of degree 4 with the property that the differential equation $y'' + Py = 0$ has a solution with n zeros, $n-2m$ of them real [10, 7, 5].

Suppose that $(b,\lambda) \in Z_J^{QES}(\mathbf{R})$. Then the corresponding eigenfunction y of (2) can be always chosen to be real. Let y_1 be a real solution of the differential equation in (2) normalized by $y_1(x) \to 0$ as $x \to +\infty$, $x \in \mathbf{R}$. Then y_1 is linearly independent of y. Consider the meromorphic function $f = y/y_1$. This function has no critical points in $\mathbf{C}$.

Consider the sectors

$$S_j = \{te^{i\theta} : t > 0,\ |\theta - \pi j/3| < \pi/6\}, \quad j = 0, \ldots, 5.$$

The subscript j in S_j will be always understood as a residue modulo 6. Function f has asymptotic values $\infty, 0, c, 0, \bar{c}, 0$ in the sectors $S_0, \ldots S_5$, where $c \in \overline{\mathbf{C}}$. It is known that f must have at least 3 distinct asymptotic values [9], so $c \neq 0, \infty$. Function f is defined up to multiplication by a non-zero real number, so we can always assume that $c = e^{i\beta}$, $0 \leq \beta \leq \pi$, where the points 0 and π can be identified. Asymptotic value c is called the *Nevanlinna parameter*. There is a simple relation between c and the Stokes multipliers [11, 8].

The sectors S_j correspond to logarithmic singularities of the inverse function f^{-1}. Thus f^{-1} has 6 logarithmic singularities that lie over four points if $c \neq \bar{c}$, or over three points if $c = \bar{c}$.

The map $(b,\lambda) \mapsto \beta$, $Z_J^{QES}(\mathbf{R}) \to \mathbf{R}$ is analytic and locally invertible [11, 2], so β can serve as a local parameter on the real QES spectral locus.

To obtain a global parametrization one needs suitable charts on $Z_J^{QES}(\mathbf{R})$, where this map is injective.

To recover f, one has to know the asymptotic value c and one more piece of information, a certain cell decomposition of the plane described below. Once f is known, b and λ are found by the formula

$$\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 = -2(z^4 - 2bz^2 + 2Jz - \lambda). \tag{3}$$

Now we describe, following [4], the cell decompositions needed to recover f from c. Suppose first that $c \notin \mathbf{R}$.

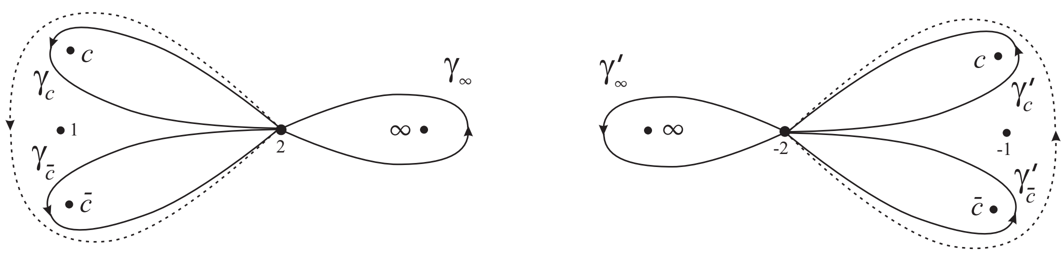

Fig. 1. Cell decompositions Φ and Φ' of the sphere.

Consider the cell decomposition Φ of the Riemann sphere $\overline{\mathbf{C}}$ shown by solid lines on the left part of Fig. 1. It consists of one vertex at the point 2, three edges (loops γ_c, $\gamma_{\overline{c}}$ and γ_∞ around non-zero asymptotic values) and four faces (cells of dimension 2). The faces are labeled by the asymptotic values $0, c, \overline{c}, \infty$. Label 0 is not shown in the picture. The face labeled 0 is the unbounded region in the picture. (The point 1 in the figure is neither a label, nor a part of the cell decomposition. It will be needed, together with the dashed line, for the limit at $\beta = 0$.) As

$$f : \mathbf{C}\backslash f^{-1}(0, \infty, c, \overline{c}) \to \overline{\mathbf{C}}\backslash\{0, \infty, c, \overline{c}\}$$

is a covering map, the cell decomposition Φ pulls back to a cell decomposition Ψ of the plane.

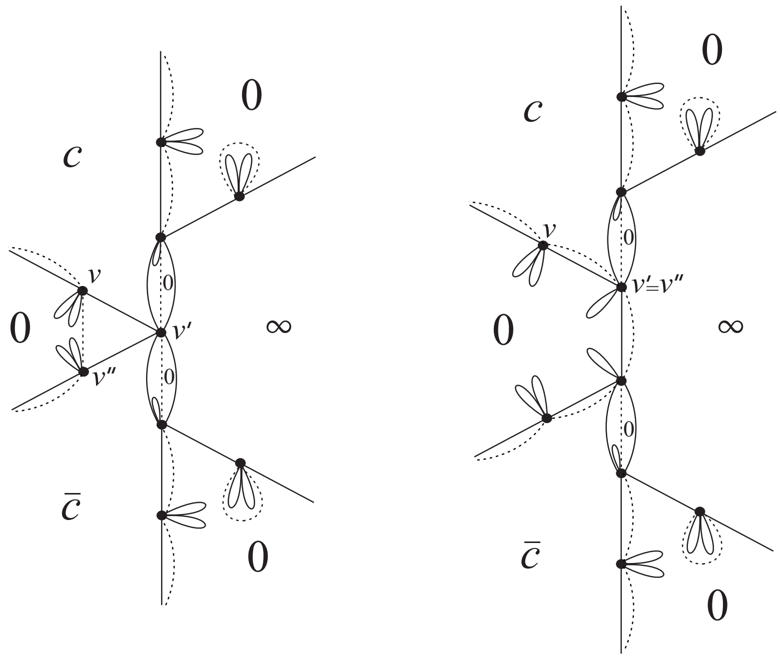

Fig. 2. Two examples of the cell decomposition Ψ of the plane (solid lines). Both eigenfunctions have two zeros, none of them real.

Two examples of Ψ are shown in Fig. 2 (solid lines). The faces of Ψ are labeled with the same labels as their images. Non-zero labels of bounded faces are omitted in the picture. The reader can restore them from the condition that labels around a vertex must be in the same cyclic order as in Fig. 1 (left, solid lines). The labeled cell decomposition Ψ defines f up to a pre-composition with an affine map of $\mathbf{C}$. Two cell decompositions define the same f if they can be obtained from each other by orientation-preserving homeomorphism of the plane.

Replacing multiple edges of the 1-skeleton of Ψ with single edges and removing the loops, we obtain a simpler cell decomposition T whose 1-skeleton is a tree, which we denote by the same letter T. The cell decomposition Ψ is uniquely recovered from its tree T embedded in the plane, [4]. The faces of T are asymptotic to the sectors S_j and the label of each face is the asymptotic value in S_j. Two faces with a common edge cannot have the same label. The cell decomposition T is invariant under the reflection in the real axis, with simultaneous interchange of c and $\overline{c}$. It is easy to classify all possible embedded trees T with labeled faces that satisfy these properties. They depend on two integer parameters k and $l \geq 0$. These trees form two families, $X_{k,l},\ k \leq 0, l \geq 0$ and $X_{k,l},\ k > 0, l \geq 0$, shown in Fig. 3.

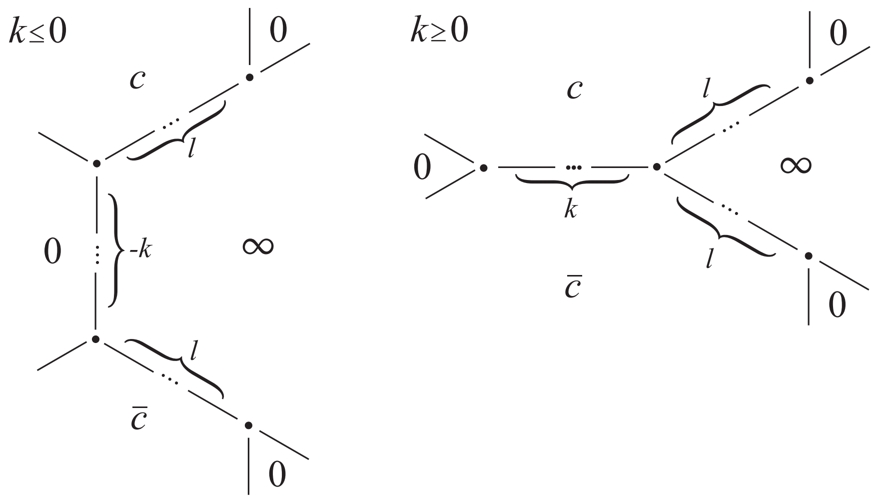

Fig. 3. Trees $X_{k,l}$.

Cell decompositions in Fig. 2 (solid lines) correspond to the trees $X_{0,1}$ and $X_{-1,1}$.

Parameters of the trees $X_{k,l}$ can be interpreted as follows:

$$k^+ := \max\{k, 0\}$$

is the number of real zeros of f, and $2l$ is the number of non-real zeros. So the total number of zeros is

$$n = 2l + k^+.$$

Functions f corresponding to trees $X_{k,l}, k \leq 0$ have $2l$ zeros, none of them real. Zeros of the eigenfunction y coincide with those of f.

For given n the number of trees $X_{k,l}$ with $k + 2l = n,\ k > 0$ is $(n+1)/2$ when n is odd, and $n/2$ when n is even.

Every tree $X_{k,l}$ and every $\beta \in (0, \pi)$ defines a meromorphic function f satisfying (3) with $J = 2l + k^+ + 1$ and some (b, λ) depending on β. This follows from a result of Nevanlinna [9], see also [4]. From this function f, the coordinates of a point (b, λ) on the real QES spectral locus are recovered by the formula (3). Thus we have a map $F : (T, \beta) \mapsto (b, \lambda)$ which we call the Nevanlinna map. This map is of highly transcendental nature: construction of f from T and β involves the uniformization theorem. We refer to [4, 5, 9] for details.

So each of the trees from our classification defines a chart of $Z^{QES}(\mathbf{R})$.

To obtain the global parametrization of $Z_J^{QES}(\mathbf{R})$, we only have to find out how these charts are pasted together.

We will see that the boundaries of our charts correspond to real values of c. When c is real, we can use instead of Φ the cell decomposition of the sphere with two loops, γ_∞ and the loop around c shown with the dashed line in the left part of Fig. 1. Taking the preimage of this cell decomposition and removing the loops and multiple edges from this preimage gives again one of the trees $X_{k,l}$.

Proof of Theorem 1. We begin with the simpler case of charts $X_{k,l},\ k > 0$. In these charts the limits as $\beta \to 0, \pi$ do not belong to the spectral locus, because the two faces labeled with c and $\overline{c}$ have a common boundary edge, so the charts defined by $X_{k,l},\ k + 2l = n, 1 \le k \le n$, make separate components of $Z_{n+1}^{QES}(\mathbf{R})$, each parametrized by $\beta \in (0, \pi)$. We call these components $\Gamma_{n,m}$ where $m = (n - k)/2$. These are simple disjoint analytically embedded curves in $\mathbf{R}^2$ parametrized by $\beta \in (0, \pi)$.

When $\beta \to 0, \pi$ we must have $b \to \pm\infty$. We'll show below that $b \to +\infty$ on both ends of $\Gamma_{n,m}$ when $k > 0$.

When n is odd (that is J is even), these curves $\Gamma_{n,m}$ parametrize the whole spectral locus $Z_n^{QES}(\mathbf{R})$.

Now consider the part of the spectral locus parametrized by $X_{k,l},\ k \le 0$. This part is present only when $n = 2l$ is even.

Lemma. *For $l \ge 0$ and $k \le 0$, we have*

$$\lim_{\beta \to \pi} F(X_{k,l}, \beta) = \lim_{\beta \to 0} F(X_{k-1,l}, \beta). \tag{4}$$

If we want β to be continuous on the spectral locus, we must discard our normalization $\beta \in [0, \pi]$ and allow β to describe the real line.

Proof of the Lemma. This is proved by the same arguments as in [5, Thm. 4.1]. When $c = \overline{c} = 1$, the cell decomposition Φ must be modified: the loops γ_c and $\gamma_{\overline{c}}$ around c and $\overline{c}$ have to be replaced by a single loop L around 1 (dashed line in Fig. 1, left part). Note that L is homotopy equivalent to the product of γ_c and $\gamma_{\overline{c}}$, and does not intersect γ_∞. Let us call the resulting cell decomposition Φ_1.

The preimage $\Psi_1 = f^{-1}(\Phi_1)$ is easy to construct from the original cell decomposition Ψ. First, removing preimages of γ_c and $\gamma_{\overline{c}}$, we obtain the cell decomposition Ψ_∞, the preimage of the loop around ∞ in Φ. Next, for each vertex v of Ψ consider the path L_v consisting of the edge of $f^{-1}(\gamma_c)$ from v to v' followed by the edge of $f^{-1}(\gamma_{\overline{c}})$ from v' to v''. Then the edge of $f^{-1}(L)$ from v to v'' is homotopic to L_v in the complement of $f^{-1}(\gamma_\infty)$. The new edges are shown with dashed lines in Fig. 2.

To deal with the case $c = \overline{c} = -1$, in a similar way, one has to choose another cell decomposition Φ' of the sphere, shown in the right part of Fig. 1 (solid lines). When $c \to -1$, Φ' collapses to Φ'_{-1} where the two loops γ'_c and $\gamma'_{\overline{c}}$ are replaced with a single loop L' around -1 (dashed line in Fig. 1, right part).

We need the transition formula from $\Psi = f^{-1}(\Phi)$ to $\Psi' = f^{-1}(\Phi')$. This formula is obtained by combining the two decompositions (see Fig. 4) and expressing the loops of Φ' in terms of the loops of Φ.

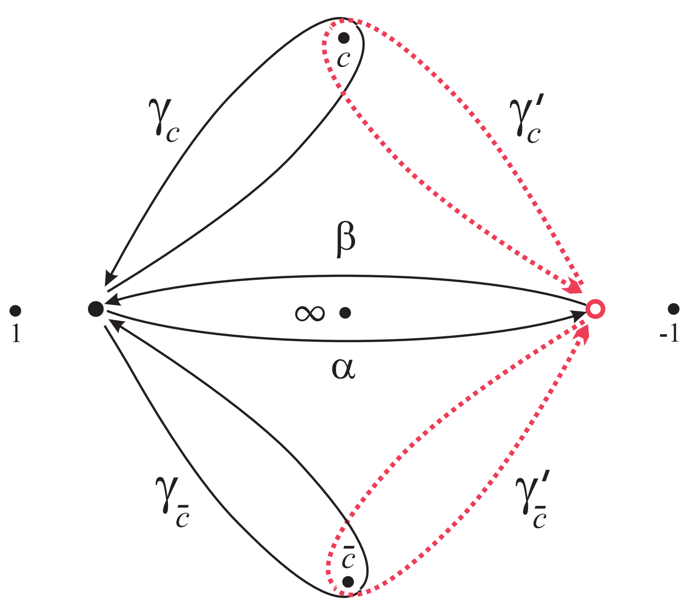

Fig. 4. Two cell decompositions of Fig. 1 combined.

The formulas using notations in Fig. 4 are:

$$\gamma_\infty = \alpha\,\beta, \quad \gamma'_\infty = \beta\,\alpha, \quad \gamma'_c = \beta\,\gamma_c\,\beta^{-1}, \quad \gamma'_{\overline{c}} = \alpha^{-1}\,\gamma_{\overline{c}}\,\alpha. \qquad (5)$$

Similar formulas were obtained in [5] in the proof of Theorem 4.1. Application of these transition formulas to the cell decomposition Ψ of the type $X_{-1,1}$ is illustrated in Figs. 5,6,7.

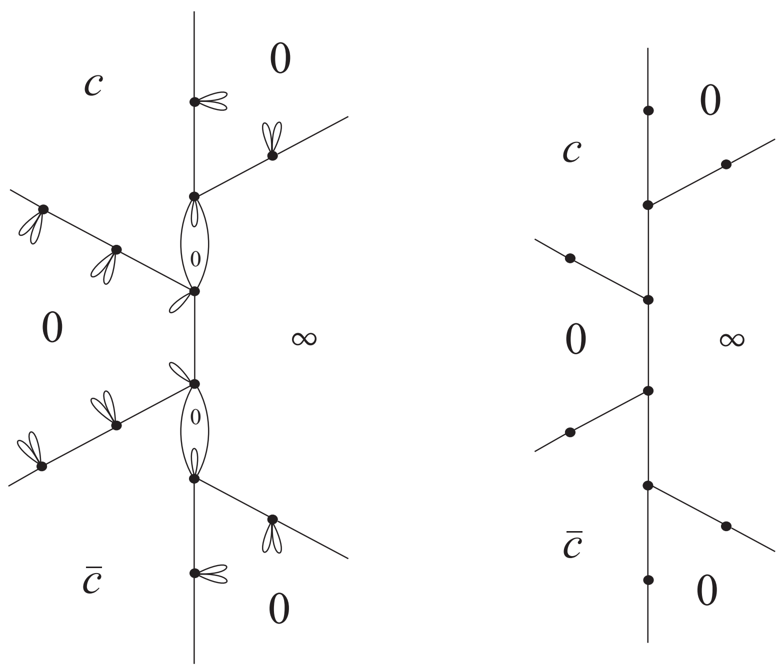

Fig. 5. Cell decomposition Ψ corresponding to the tree $X_{-1,1}$.

In Fig. 5, we use Φ from Fig. 3. The cell decomposition Ψ (left) corresponds to the tree $X_{-1,1}$ (right).

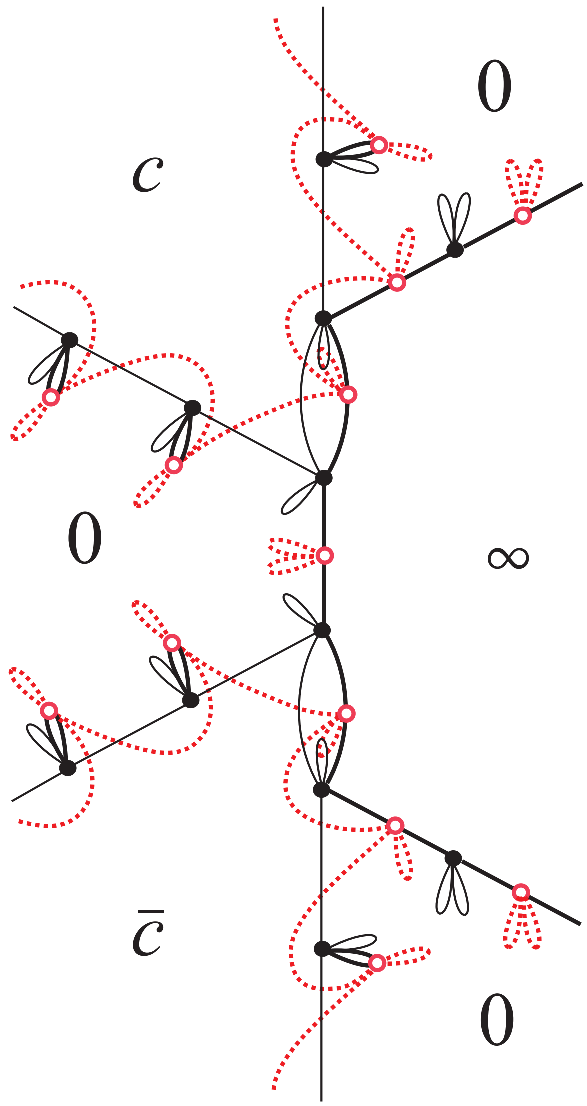

Fig. 6. Passing from Ψ to Ψ'.

In Fig. 6, the circles denote the vertices of Ψ' (preimages of the vertex of Φ') and the dotted lines correspond to the preimages of γ'_c and $\gamma'_{\bar c}$ determined from (5). The preimages of γ_∞ and γ'_∞ coincide. They are shown with the bold solid line. Removing the preimages of γ_c and $\gamma_{\bar c}$ (thin solid lines in Fig. 6) and the vertices of Ψ, we obtain the cell decomposition Ψ' shown in Fig. 7 (left) corresponding to the tree $X_{-2,1}$ (right).

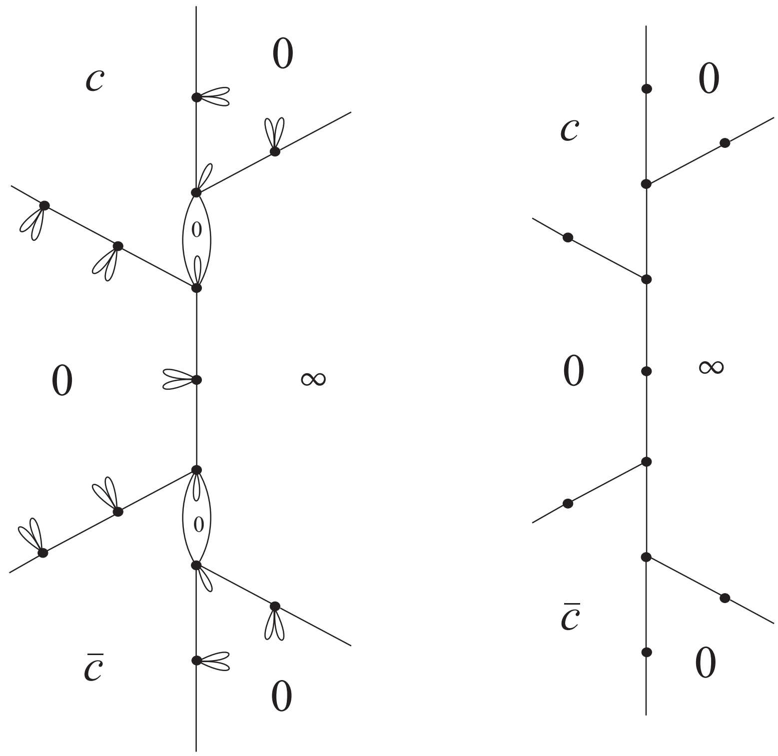

Fig. 7. Cell decomposition Ψ' corresponding to the tree $X_{-2,1}$.

Thus, for $k \leq 0$, the cell decomposition of the plane obtained from $X_{k,l}$ in the limit $\beta \to 0$ as the preimage of Φ_1 is equivalent to the cell decomposition obtained from $X_{k-1,l}$ in the limit $\beta \to \pi$ as the preimage of Φ_1'. Since $\Phi_1' = -\Phi_1$, Nevanlinna theory implies that the corresponding functions f and f' satisfy $f' = -f$. Hence these two functions correspond to the same point of $Z^{QES}(\mathbf{R})$. This completes the proof of Lemma.

Now we continue the proof of Theorem 1.

For even $n = 2l$ each chart $X_{k,l},\ k \leq 0$ parametrizes one curve in the real QES spectral locus, and we call this curve $\Gamma_{n,n/2}$. We parametrize the curve $\Gamma_{n,n/2}$ by the real line, so that the number k increases, thus the right end of $\Gamma_{n,n/2}$ is parametrized by the chart $X_{0,n/2}$. So when the parameter $t \in \mathbf{R}$ on $\Gamma_{n,n/2}$ tends to $+\infty$, the asymptotic value $c = \exp(i\beta)$ tends to 1. On the other hand, when $t \to -\infty$ on $\Gamma_{n,n/2}$ the asymptotic value c does not have a limit; it oscillates, passing each point of the unit circle infinitely many times.

The curves $\Gamma_{n,m}$ are disjoint. Indeed, different cell decompositions give different functions f. This proves the first two statements of Theorem 1.

Now we deal with asymptotic behavior of our curves $\Gamma_{n,m}$. We use the rescaling of (2) as in [6]. The QES spectral locus is defined by a polynomial equation $Q_{n+1}(b, \lambda) = 0$ which is of degree $n+1$ in λ. So on a ray $b > b_0$ there are $n = 1$ branches $\lambda_j(b)$. In [6, Eq. (25)], we found that all λ_j have asymptotics $\lambda(b) \sim b^2 + O(\sqrt{b}),\ b \to \infty$, and as $b \to +\infty$, each QES eigenfunction y_j tends to some eigenfunction Y_ℓ of the harmonic oscillator

$$-Y'' + 4z^2 Y = \mu Y, \quad Y(it) \to 0, \quad t \to \pm\infty. \tag{6}$$

The eigenvalues of this harmonic oscillator are $\mu_\ell = 2(2\ell{+}1), \ldots, \ell = 0, 1, \ldots.$

Only one of the eigenfunctions y_j can tend to a given Y_ℓ, and the corresponding eigenvalue satisfies

$$\lambda_j(b) = b^2 + (\mu_\ell - 2J + o(1))\sqrt{b}, \quad b \to +\infty.$$

It follows that all λ_j are real. The graph of each λ_j is a part of a curve $\Gamma_{n,m}$, and each $\Gamma_{n,m}$ has only two ends. So each $\Gamma_{n,m}$ contains at most two graphs of λ_j. The total number of these graphs λ_j is $n+1$ and the total number of curves $\Gamma_{n,m}$ is $(n{+}1)/2$ when n is odd, and $n/2{+}1$ when n is even. It follows that each $\Gamma_{n,m}$ contains two graphs λ_j when n is odd, and each $\Gamma_{n,m}$ except one contains two λ_j, and the exceptional one, $\Gamma_{n,n/2}$ contains one graph of λ_j, when n is even.

Thus $b \to +\infty$ as $c \to \pm 1$ in the $X_{k,l}$-charts with $k > 0$, which proves the third statement of the Theorem. To prove the last statement for $X_{k,l}$-charts with $k > 0$, we study the zeros of the eigenfunctions.

The eigenfunction Y_ℓ of (6) corresponding to the eigenvalue μ_ℓ has exactly ℓ zeros on $i\mathbf{R}$ and no other zeros in $\mathbf{C}$. One of these zeros is real iff ℓ is odd.

When $b \to +\infty$, each QES eigenvalue $\lambda_j(b)$ must tend to some μ_ℓ, and the corresponding QES eigenfunction tends to Y_ℓ. Suppose that the tree corresponding to $\lambda(b)$ is $X_{k,l},\ k > 0$; we are going to find the correspondence between ℓ and k, l.

We have $c \to \pm 1$, as $b \to +\infty$, so the tree $X_{k,l}$ must collapse in the limit: all edges separating a face labeled c and a face labeled $\overline{c}$ must be erased, and the resulting tree must be a tree corresponding to some Y_ℓ (Fig. 8). The trees corresponding to Y_ℓ are constructed similarly to those corresponding to y, using the two loop cell decomposition of the sphere, consisting of γ_∞ and the dashed loop in Fig. 1, left.

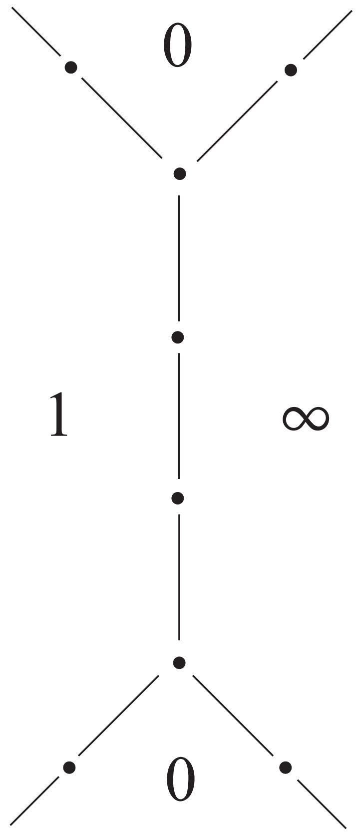

Fig. 8. The tree corresponding to Y_3.

For general results on convergence of Nevanlinna functions like our f we refer to [12]. By counting the zeros whose limiting position is on $i\mathbf{R}$, we conclude that this can happen exactly when $\ell = l$ for even ℓ, and $\ell = l + 1$ for odd ℓ. So, for fixed n and m, there are two values of ℓ. These two values correspond to $c \to \pm 1$.

Now we consider the degeneration of the $X_{0,l}$ chart with $l \geq 0$, the chart which parametrizes the left end of $\Gamma_{n,n/2}$, $n = 2l$. On the left end of $\Gamma_{n,n/2}$, where in our parametrization described after the Lemma, $t \to -\infty$, there are infinitely many points $\Gamma_{n,n/2}(t_k)$ which belong to the real QES locus, and where the asymptotic value c is real. It was proved in [6] that these are exactly those points where $Z_J^{QES}(\mathbf{R})$ crosses the non-quasi-exactly solvable part of $Z_J(\mathbf{R})$, and these points correspond to $b_k \to -\infty$.

It follows that it is the right end of $\Gamma_{n,n/2}$ on which $b \to +\infty$. Inspection of the cell decomposition corresponding to this right end shows that the limit eigenfunction has n zeros. So it tends to Y_n.

So the ordering of the ends of the curves $\Gamma_{n,m}$ corresponds to the natural ordering of the first $n+1$ eigenvalues of the harmonic oscillator. This completes the proof.

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Department of Mathematics
Purdue University
West Lafayette, IN 47907
eremenko@math.purdue.edu
agabriel@math.purdue.edu