Trace identities

March 18, 2023

1. We denote similar matrices by $A \sim B$. Notice that $ABC \sim BCA$. Therefore

$$\mathrm{tr}\,(ABC) = \mathrm{tr}\,(BCA),$$

but in general it is not true that $\mathrm{tr}\,(AB) = \mathrm{tr}\,(BA)$.

Similarly we obtain that trace of the product does not change under cyclic permutations.

2. Let $A \in SL(2)$, so that $\det A = 1$. We have

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{and} \quad A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

so

$$A + A^{-1} = (\mathrm{tr}\,A)I,$$
$$AB + A^{-1}B = (\mathrm{tr}\,A)B,$$

so

$$\mathrm{tr}\,(AB) + \mathrm{tr}\,(A^{-1}B) = \mathrm{tr}\,A\,\mathrm{tr}\,B.$$

If $A = B$ we obtain

$$\mathrm{tr}\,A^2 + 2 = (\mathrm{tr}\,A)^2.$$

3. Prove

$$\mathrm{tr}\,(ABA^{-1}B^{-1}) = (\mathrm{tr}\,A)^2 + (\mathrm{tr}\,B)^2 + (\mathrm{tr}\,(AB))^2 - \mathrm{tr}\,A\,\mathrm{tr}\,B\,\mathrm{tr}\,(AB) - 2.$$

(Mumford et al., p. 189)

4. Prove

$$\mathrm{tr}\,(ABC) + \mathrm{tr}\,(BAC) = \mathrm{tr}\,(A)\mathrm{tr}\,(BC) + \mathrm{tr}\,(B)\mathrm{tr}\,(AC) + \mathrm{tr}\,(C)\mathrm{tr}\,(AB) - \mathrm{tr}\,(A)\mathrm{tr}\,(B)\mathrm{tr}\,(C).$$

(H. Vogt, S. 7)

5. Prove

$$\mathrm{tr}\,(A^n) = \mathrm{tr}\,(A)^n - n\mathrm{tr}\,(A)^{n-1} + \frac{n(n-3)}{1.2}\mathrm{tr}\,(A)^{n-2} - \dots,$$

epansion of $(\omega + 1/\omega)^n$ in powers of $\omega + 1/\omega$.