Traces of elements of the modular group

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Let
$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}.$$
These two matrices generate the free group which is called $\Gamma(2)$, the principal congruence subgroup of level 2.

With arbitrary integers $n_j \neq 0$, consider the trace of the product
$$p(n_1, \ldots, n_{2k}) = \mathrm{tr}(A^{n_1} B^{n_2} \ldots B^{n_{2k}}).$$
It is easy to see that p is a polynomial in $2k$ variables with integer coefficients. This polynomial can be written explicitly though the formula is somewhat complicated.

Choosing an arbitrary sequence σ of $2k$ signs $\pm$, we make a substitution
$$q_\sigma(x_1, x_2, \ldots, x_{2k}) = p(\pm(1 + x_1), \pm(1 + x_2), \ldots, \pm(1 + x_{2k})).$$
We conjecture that for every σ, all coefficients of this polynomial q_σ are of the same sign, that is the sequence of coefficients of q_σ has no sign changes.

This has been verified by symbolic computation for small k.